An extended class of dissipative systems

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ABSTRACT: Earlier results on algebraic criteria for dissipative systems are extended to a wider class of supply rates.

1. Introduction

The property of being a dissipative system is always relative to some specified energy input function. At one extreme, we can define input-output dissipativeness via the inequality

$$E(0, u, y, t_0, t_1) \geq 0$$

for all u and all $t_1 \geq t_0$, where $E(x_0, u, y, t_0, t_1)$ is the energy input over the time interval $[t_0, t_1)$ resulting from initial state x_0 and input u, where y is the system output. It turns out – see section 2 – that one can get some results with only the mildest constraints on the energy function E. At this level of abstraction, though, the results are not particularly helpful in terms of ease of calculation.

We get stronger results with special forms of the function E. In particular, there are known algebraic criteria [1, 3] when E is the integral

$$E(x_0, u, y, t_0, t_1) = \int_{t_0}^{t_1} (y(t)^T Q y(t) + 2y(t)^T S u(t) + u(t)^T R u(t))\, dt$$

and Q, S, and R are constant matrices.

In the present paper, we want to extend this to the case where the integrand is not necessarily a quadratic function of the output. That is, we look at the case

$$E(x_0, u, x, t_0, t_1) = \int_{t_0}^{t_1} (q(x(t)) + 2s(x(t))^T u(t) + u(t)^T R(x) u(t))\, dt \tag{1}$$

where $q(\cdot)$ and $s(\cdot)$ are more arbitrary nonlinear functions of the state. The integrand continues to be quadratic in the input. The state equations are of the form

$$\frac{dx}{dt} = f(x) + G(x)u \tag{2}$$

$$y = x$$

That output equation is one limitation of our analysis. It is easy to see that we could trivially extend the analysis to an output equation

$$y = h(x)$$

but not as far as allowing an input term in the output equation. To keep the notation simple, then, we will restrict discussion to the case where the output is the whole state.

To avoid complications, we shall also assume that the state space is completely reachable, and assume sufficient smoothness to give differentiable storage functions.

2. Some known results

As a starting point, we can define dissipativeness either by the input-output condition

$$E(0, u, y, t_0, t_1) \geq 0$$

or by the state-space condition

$$\phi(x(t_0)) + E(x(t_0), u, y, t_0, t_1) \geq \phi(x(t_1))$$

where in the latter case we require the storage function ϕ to satisfy $\phi(0) = 0$ and $\phi(x) \geq 0$ for all x. This latter condition was the original definition of a dissipative system given by Willems [4]. It was later shown [2, 3] that, given reachability of the state space, the two definitions are equivalent.

Let us define the *available storage* as

$$\phi_a(x_0) = -\inf_u E(x_0, u, y, t_0, t_1)$$

with boundary conditions $x(t_0) = x_0,\ x(t_1)$ unconstrained.

We can also define a *required supply*

$$\phi_r(x_0) = \inf_u E(0, u, y, t_{-1}, t_0)$$

with boundary conditions $x(t_{-1}) = 0$, $x(t_0) = x_0$.

Then it turns out that every storage function ϕ satisfies

$$0 \leq \phi_a(x) \leq \phi(x) \leq \phi_r(x)$$

for all x. It also turns out that the set of all storage functions is a convex set.

3. A preliminary calculation

Suppose that we know that a scalar quadratic function of a vector u is nonegative for all u. That is,

$$u^T A u + 2b^T u + c \geq 0$$

where, without any loss of generality, the matrix A can be assumed to be symmetric. (The 2 is introduced to simplify the algebra.) We want to show that this can be simplified down to the form

$$(\ell + Wu)^T(\ell + Wu) \geq 0$$

This was already shown in [3], but since the proof is not entirely obvious it seems like a good idea to give a more detailed proof.

Clearly we must have $A \geq 0$. That means that there is an orthonormal matrix T such that

$$TAT^T = \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix}$$

and D is positive definite. Let Tu and Tb be partitioned as

$$Tu = \begin{bmatrix} x \\ y \end{bmatrix} \qquad\qquad Tb = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Then the original inequality becomes

$$x^T Dx + 2b_1^T x + 2b_2^T y + c \geq 0$$

or

$$\left(D^{1/2}x + D^{-1/2}b_1\right)^T \left(D^{1/2}x + D^{-1/2}b_1\right) - b_1^T D^{-1} b_1 + 2b_2^T y + c \geq 0$$

This is supposed to be true for all x and y, which is possible only if $b_2 = 0$. Also, by checking the case $x = -D^{-1}b_1$, we can see that $c - b_1^T D^{-1} b_1 \geq 0$. Factoring that last quantity as $k^T k$, we get

$$\left(D^{1/2}x + D^{-1/2}b_1\right)^T \left(D^{1/2}x + D^{-1/2}b_1\right) + k^T k \geq 0$$

which can also be written as

$$(\ell + Wu)^T (\ell + Wu) \geq 0$$

where

$$\ell = \begin{bmatrix} D^{-1/2}b_1 \\ k \end{bmatrix} \qquad\qquad W = \begin{bmatrix} D^{1/2} & 0 \\ 0 & 0 \end{bmatrix} T$$

The factorisation is not unique, but this is probably the simplest choice of ℓ and W.

4. The main result

Theorem. The system (2) is dissipative with respect to the E defined by equation (1) iff there exist functions $\phi(x)$, $\ell(x)$, and $W(x)$ such that $\phi(x) = 0$, $\phi(x) \geq 0$ for all x, and

$$\nabla\phi^T f = q - \ell^T \ell$$

$$\frac{1}{2} G^T \nabla\phi = s - W^T \ell$$

$$R = W^T W$$

Proof. If we have a function $\phi(x)$ such that

$$\phi\left(x(t_0)\right) + \int_{t_0}^{t_1} (q(x(t)) + 2s(x(t))^T u(t) + u(t)^T R(x(t)) u(t))\, dt \geq \phi\left(x(t_1)\right)$$

we can reduce it to the differential form

$$-\frac{d\phi\left(x(t)\right)}{dt} + q(x(t)) + 2s(x(t))^T u(t) + u(t)^T R\left(x(t)\right) u(t) \geq 0$$

or

$$-\nabla\phi(x)^T f(x) - \nabla\phi(x)^T G(x) u + q(x) + 2s(x)^T u + u^T R(x) u \geq 0$$

For any fixed x this is a quadratic function of u. By the argument of section 3, we can find $W(x)$ and $\ell(x)$ such that

$$-\nabla\phi(x)^T f(x) - \nabla\phi(x)^T G(x)u + q(x) + 2s(x)^T u + u^T R(x)u$$
$$= \left(W(x)u + \ell(x)\right)^T \left(W(x)u + \ell(x)\right)$$

Which must be true for any fixed x. Expanding this out, and equating powers of u, we get

$$\nabla\phi^T f = q - \ell^T \ell$$
$$\frac{1}{2} G^T \nabla\phi = s - W^T \ell$$
$$R = W^T W$$

This proves half of the result. For the other half, we just have to start with these three equations and work backwards through the same argument. □

It should be obvious that we can also write these equations as an inequality

$$\begin{bmatrix} q - \nabla\phi^T f & s - \frac{1}{2} G^T \nabla\phi \\ s^T - \frac{1}{2}\nabla\phi^T G & R \end{bmatrix} \geq 0$$

In general the solutions are non-unique. In fact, not even the number of rows of W is fixed.

5. Conclusions

The known result in [1, 3] is based on an energy function where the integrand is jointly quadratic in both the input and the output. We have now shown that an almost identical result can be found if we relax that condition to requiring only that the integrand be quadratic in the input.

References

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