On a condition needed for the supply rate for dissipative systems

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ABSTRACT: Some results in dissipative systems theory require that the supply rate be such that it can be forced negative by choice of input. In this note we show that this can be expressed as an eigenvalue condition.

1. Introduction

The theory of dissipative systems [Wil72] [HM80] [Moy14] defines a system to be dissipative if a certain quantity that depends on the input and output is nonnegative for all possible inputs. In the most general case, the quantity in question is an abstract “input energy”. Most commonly we restrict ourselves to time-invariant continuous-time systems and $\mathcal{L}_2$ signal spaces, in which case we express the “energy” as a time integral. This leads to the following definitions.

Definition 1. A system with input *u* and output *y* is dissipative, in the input-output sense, with respect to supply rate *w* iff

$$\int_0^T w(u(t), y(t)\, dt \geq 0 \tag{1}$$

for all $T \geq 0$ and all inputs u.

Definition 2. A system with state *x*, input *u*, and output *y* is dissipative, in the state-space sense, with respect to supply rate *w* iff there exists a function $\varphi(x)$, with $\varphi(0) = 0$ and $\varphi(x) \geq 0$ for all *x*, and $\varphi(x)$ finite for all reachable *x*, such that

$$\varphi(x(0) + \int_0^T w(u(t), y(t)\, dt \geq \varphi(x(T)$$

for all $T \geq 0$ and all inputs u.

The function $\varphi(\cdot)$ is called a storage function, because it is analogous to stored energy.

If a state-space description exists, then for the purposes of comparing the two definitions it is usual to require that the initial state be zero in Definition 1. In that case, an important conclusion of the theory [HM80] is that the two definitions are equivalent.

Not all supply rates $w(\cdot,\cdot)$ are equally useful. Consider, for example, a supply rate with the property $w(u, y) \geq 0$ for all u and y. In that case, inequality (1) will always be satisfied, so that *every* system is dissipative with respect to that supply rate. Clearly, this is not a useful property. To make dissipativeness useful, we need to permit the supply rate to go negative, so that (1) defines a property of the system rather than simply of w.

Because of this, a class of "interesting" supply rates was introduced in [HM80].

Property A. For any $y \neq 0$, there exists a $u(y)$ such that $w(u(y), y) < 0$.

It is important to understand that this does not contradict inequality (1). In (1), *u(t)* and *y(t)* are not independent variables; they are linked by the dynamics of the system under consideration. In Property A, we are looking at *w* as a function of two variables, with no constraint introduced by the system.

The assumption of Property A has proven to be useful in establishing several results. In [HM80], it turned out to be the main condition needed to ensure that the storage function is positive definite. In [Moy75] it was shown to be a condition that allowed establishing a link between time-domain and frequency-domain criteria in linear-quadratic control theory. In [Moy14, chapter 8], which deals with frequency domain conditions for dissipativeness, it turns out to be the assumption needed to link behaviour on the $j\omega$ axis to behaviour elsewhere on the complex plane.

2. Quadratic supply rates

The purpose of this note is to derive a condition that implies Property A. We cannot do this for completely general supply rates, but we can do so in the widely used special case of a quadratic supply rate. Let us therefore confine our attention to

$$w(u, y) = y^T Q y + 2y^T S u + u^T R u = \begin{bmatrix} y^T & u^T \end{bmatrix} \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

where *Q*, *S*, and *R* are constant matrices of appropriate size.

In what follows, it will be important to note that the matrix

$$M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$$

is a real symmetric matrix. That means that it has real eigenvalues.

3. The main result

Suppose that the system of interest has *m* scalar inputs and *n* scalar outputs. That is, *u* is an *m*-vector and *y* is an *n*-vector.

Theorem. A sufficient condition for Property A to hold is that *M* have at least *n* negative eigenvalues.

4. Proof of the result

Since *M* is real and symmetric, it can be diagonalised by an orthogonal matrix. That is, there exists a real matrix *V*, with $V^{-1} = V^T$, such that

$$VMV^T = \Lambda$$

where Λ is diagonal. We can choose V such that the negative eigenvalues of *M* come first in Λ, then the zero eigenvalues, and then the positive ones. Then we have

$$VMV^T = \begin{bmatrix} -\Lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Lambda_2 \end{bmatrix}$$

where Λ_1 and Λ_2 are positive definite diagonal matrices. In some cases – for example, when all of the eigenvalues are negative – some of these blocks are missing.

Of course, the partitioning is not the same as the partitioning of *M*, but if *M* has at least *m* negative eigenvalues then we have

$$M = V^T \begin{bmatrix} -\Lambda_\mathrm{a} & 0 \\ 0 & \Lambda_\mathrm{b} \end{bmatrix} V$$

where Λ_a consists of the first *n* rows and columns of Λ_1, and Λ_b holds everything else. In general the diagonal entries of Λ_b will have a mixture of signs.

Let *V* be partitioned in the same way as *M*, as

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$$

and define

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} V_{11}y + V_{12}u \\ V_{21}y + V_{22}u \end{bmatrix}$$

Observe that

$$w(u, y) = z^T \begin{bmatrix} -\Lambda_\mathrm{a} & 0 \\ 0 & \Lambda_\mathrm{b} \end{bmatrix} z = -z_1^T \Lambda_a z_1 + z_2^T \Lambda_b z_2$$

so we will have the desired result if we can show that the z_1 term dominates.

Consider first the case where V_{22} is nonsingular. In that case it is easy to show that $V_{11} - V_{12}V_{22}^{-1}V_{21}$ is also nonsingular. Let $u = -V_{22}^{-1}V_{21}y$, so that $z_2 = 0$ and $z_1 = (V_{11} - V_{12}V_{22}^{-1}V_{21})y$. Nonsingularity of that last coefficient matrix means that $z_1 \neq 0$ whenever $y \neq 0$, which implies Property A.

At the other extreme, consider the case where $V_{22} = 0$. In this case, it turns out that any sufficiently large u will make $w(u, y) < 0$. The fact that V is a unitary matrix means that

$$V^T V = \begin{bmatrix} V_{11}^T & V_{21}^T \\ V_{12}^T & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & 0 \end{bmatrix} = \begin{bmatrix} V_{11}^T V_{11} + V_{21}^T V_{21} & V_{11}^T V_{12} \\ V_{12}^T V_{11} & V_{12}^T V_{12} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and therefore $V_{12}^T V_{12} = I$ and $V_{12}^T V_{11} = 0$. The equations for z_1 and z_2 now become

$$z_1 = V_{11}y + V_{12}u$$
$$z_2 = V_{21}y$$

Let $u = \alpha k$, where α is a scalar and k is an arbitrary nonzero vector. Now $z_1 = V_{11}y + \alpha V_{12}k$, and the fact that $V_{12}^T V_{12} = I$ means that $\alpha V_{12}k \neq 0$ whenever $k \neq 0$. For any fixed given y, z_2 is a constant and z_1 is a linear function of α plus a constant. That means that $w(u, y)$ will be negative for sufficiently large α.

The remaining case, and unfortunately the most complicated case, is where V_{22} is singular but nonzero. Because of the tedious algebra, the proof for this case has been relegated to the appendix.

5. Conclusions

Property A is an important assumption for several aspects of dissipative systems theory, but until now the task of expressing this property in terms of the (Q,S,R) matrices has proved to be elusive. Now we have finally been able to express this as an eigenvalue condition.

The present result is a sufficient but not a necessary condition for Property A to hold. A necessary condition is as yet not available.

References

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Appendix 1: The case of singular V_{22}

To illustrate how the proof works, consider first the case where V_{22} is in block form

$$V_{22} = \begin{bmatrix} V_3 & 0 \\ 0 & 0 \end{bmatrix}$$

where V_3 is nonsingular. Now we have

$$V^T V = \begin{bmatrix} V_{11}^T & V_{21a}^T & V_{21b}^T \\ V_{12a}^T & V_3^T & 0 \\ V_{12b}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12a} & V_{12b} \\ V_{21a} & V_3 & 0 \\ V_{21b} & 0 & 0 \end{bmatrix} = \begin{bmatrix} \times & \times & V_{11}^T V_{12b} \\ \times & \times & V_{12a}^T V_{12b} \\ V_{12b}^T V_{11} & V_{12b}^T V_{12a} & V_{12b}^T V_{12b} \end{bmatrix}$$

where the $\times$ entries indicate subblocks whose precise values are irrelevant to our argument. Since this must be equal to a unit matrix, we have inter alia that $V_{12b}^T V_{12b} = I$. That means that V_{12b} has linearly independent columns.

The equations for z are

$$z_1 = V_{11} y + V_{12a} u_a + V_{12b} u_b$$

$$z_{2a} = V_{21a} y + V_3 u_a$$

$$z_{2b} = V_{21b} y$$

where z_2 and u have been partitioned in the obvious way. Now, for any given y, set $u_a = V_3^{-1} V_{21a} y$ and $u_b = \alpha k$, where α is a scalar and k is an arbitrary nonzero vector. Now $z_1 = V_{11} y + \alpha V_{12b} k$, and the fact that V_{12b} has full column rank means that $V_{12b} k \neq 0$ whenever $k \neq 0$.

Since, for each fixed y, z_2 is constant while z_1 can be made arbitrarily large by choice of α, we conclude that $w(u, y) < 0$ for sufficiently large α.

In the more general case where V_{22} is singular but nonzero, there exist nonsingular matrices T_1 and T_2 such that

$$T_1 V_{22} T_2 = \begin{bmatrix} V_3 & 0 \\ 0 & 0 \end{bmatrix}$$

where V_3 is nonsingular. The z equations are now

$$z_1 = V_{11} y + V_{12} u$$

$$z_2 = V_{21} y + T_1^{-1} \begin{bmatrix} V_3 & 0 \\ 0 & 0 \end{bmatrix} T_2^{-1} u$$

Let $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = T_2^{-1} u$ and $\begin{bmatrix} z_{2a} \\ z_{2b} \end{bmatrix} = T_1 z_2$. Also let $V_{12} T_2$ and $T_1 V_{21}$ be partitioned as $V_{12} T_2 = [V_{12a} \quad V_{12b}]$ and $T_1 V_{21} = \begin{bmatrix} V_{21a} \\ V_{21b} \end{bmatrix}$. Then

$$z_1 = V_{11} y + V_{12a} v_1 + V_{12b} v_2$$

$$z_{2a} = V_{21a} y + V_3 v_1$$

$$z_{2b} = V_{21b} y$$

As before, set $v_1 = V_3^{-1} V_{21a} y$, which makes $T_1 z_2$ (and therefore z_2) depend only on y. Also set $v_2 = \alpha k$, where α is a scalar and k is an arbitrary constant vector. If we can show that $V_{12b} k \neq 0$, then by choice of α we can make $w(u, y) = w(T_2 v, y) < 0$.

Suppose the contrary. If $V_{12b}k = 0$ for some $k \neq 0$, then we have

$$V_{12}T_2 \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} V_{12a} & V_{12b} \end{bmatrix} \begin{bmatrix} 0 \\ k \end{bmatrix} = 0$$

That means that we have a $b = T_2 \begin{bmatrix} 0 \\ k \end{bmatrix} \neq 0$ such that $V_{12}b = 0$. We also have

$$T_1V_{22}T_2 \begin{bmatrix} 0 \\ k \end{bmatrix} = \begin{bmatrix} V_3 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ k \end{bmatrix}$$

$$T_1V_{22}b = 0$$

which implies $V_{22}b = 0$. Thus we have

$$\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \begin{bmatrix} 0 \\ b \end{bmatrix} = 0$$

which is impossible because $b \neq 0$ and V is nonsingular. We therefore conclude that $V_{12b}k \neq 0$ for all $k \neq 0$.