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A NEW LYAPUNOV FUNCTION FOR INTERCONNECTED POWER SYSTEMS

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ABSTRACT

A new Lyapunov function is constructed for a general n-machine power system with non-zero transfer conductance. The theory of dissipative dynamic systems is used and the procedure calls for checking positive definiteness of a sparse matrix $\hat{Q}$. In this paper we report the results only for the case of non-uniform damping which is more complex than the uniformly damped system. The procedure is iterative and may fail to converge for very small damping and strong interconnections. The Lyapunov Function is a quadratic term plus a weighted sum of integrals a form not reported before. This is an extension of the previously reported [2] quadratic Lyapunov function.

1. Introduction

This paper reports a successful procedure to construct Lyapunov functions for a multimachine power system, with transfer conductances included, using dissipative systems theory [5] for large-scale interconnected system [4]. The stability test consists in testing a sparse matrix $\hat{Q}$ for positive definiteness. The derived Lyapunov Fucntion, a quadratic term plus a weighted sum of integrals of all the non linearities is of a form not found in the literature to date.

Araki et.al.[7] and Jocic et.al.[8] derived Lyapunov functions with only (n-1) integral terms for an n-machine system. The method in [7] works only for the uniformly damped power system. There is an extension of [8] for the nonuniformly damped system. The method presented in this paper gives a larger region of stability and less restrictive conditions on system parameters for the existence of a Lyapunov function. All the attempts to derive a Lyapunov function in the manner of Aylett, with path independent integral terms, have failed [10], although there exist some approximate methods [1],[11]. To obtain better results we manipulate system equations to get the results for some sector [o,k], k = diag (k_i) and $o \le k_i \le 1$. This manipulation is possible because we decompose the system into small subsystems so the computational requirements increase only linearly.

The paper is organised as follows: Section 2 discusses the problem formulation and Section 3 introduces the dissipative systems theory for interconnected systems. Section 4 introduces an n-machine power system first and then gives a step-by-step procedure to construct a Lyapunov function. Section 5 demonstrates the procedure using a numerical example and Section 6 concludes the paper. In this paper we have discussed power system with non uniform damping only. The case of uniform damping being easier of the two is omitted and can be found in reference [3].

2. Dynamical System

We will study the stability of the following system (1a) and (1b). The system (1) is an interconnection of m linear subsystems (1a) and $m^2 + 2m$ nonlinear dynamic subsystems (1b).

$$\begin{bmatrix} \dot{x}_i \\ \dot{x}_n \\ \dot{z}_i \end{bmatrix} = \begin{bmatrix} \lambda_i & o & -\mu_{i1} \\ o & -\lambda_n & \mu_{i2} \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_i \\ x_n \\ z_i \end{bmatrix} + \begin{bmatrix} 1 & o \\ o & 1 \\ o & o \end{bmatrix} \begin{bmatrix} u_{i1} \\ u_{i2} \end{bmatrix} \tag{1a}$$

$$y_i = \begin{bmatrix} 1 & -1 & \beta \end{bmatrix} \begin{bmatrix} x_i \\ x_n \\ z_i \end{bmatrix} \quad i = 1,..,m$$

for $k = 1,\dots,m^2 + 2m$

$i,j = 1,\dots,m$

$$\dot{\sigma}_k = -\beta\sigma_k + u_{k+m} \tag{1b}$$

$$y_{m+k} = \begin{cases} \Psi_{ij}(y_i - y_j) \quad , & k = m(i-1)+j \text{ and } i \neq j \\ \Psi_{in}(y_i), k = m(i-1)+i & \\ \Psi_{ij_1}(y_j), k = m^2 + j & \\ \Psi_{nj}(y_j), k = m^2 + m + j & \end{cases}$$

$$\text{where} \quad u_{i1} = -b_{in}\,\psi_{in}(\sigma_i) - \sum_{\substack{j=1 \\ j \neq i}}^{m} b_{ij}\psi_{ij}(\sigma_i - \sigma_j)$$

$$u_{i2} = b_{ni}\psi_{ni_1}(\sigma_i) + \sum_{\substack{j=1 \\ j \neq i}}^{m} b_{nj}\psi_{nj}(\sigma_j)$$

$$u_{k+m} = \begin{cases} x_i - x_j + \beta(z_i - z_j), & z \neq j \text{ and } k = m(i-1)+j \\ x_i + \beta z_i, & k = m(i-1)+i \\ x_j + \beta z_j, & k = m^2 + j \\ x_j + \beta z_j, & k = m^2 + m + j \end{cases}$$

- $n = m + 1$
- λ_i, b_{ij} are positive constraints
- each of the $m^2 + 2m$ nonlinearity $\psi_{ij}(\cdot)$ lies in some sector [o,k].

The system (1) is obtained by using the multiplier $(s + \beta)$ along with the following system (2). It can be shown that the stability of system (1) implies the stability of system (2). The use of multipliers to enhance the applicability of theorems giving the stability limits for nonlinear systems is well known [5].

$$\begin{aligned}
\dot{x}_i &= -\lambda_i x_i - \mu_{i1} z_i + u_{i1} \\
\dot{x}_n &= -\lambda_n x_n + \mu_{i2} z_i + u_{i2} \\
\dot{z}_i &= x_i - x_n \qquad\qquad i = 1,\dots,m
\end{aligned} \tag{2}$$

where

$$u_{i1} = b_{ni}\Psi_{in}(z_i) - \sum_{\substack{j=1 \\ j\neq i}}^{m} b_{ij}\Psi_{ij}(z_i - z_j)$$

$$u_{i2} = b_{ni}\Psi_{ni_i}(z_i) + \sum_{\substack{j=1 \\ j\neq i}}^{m} b_{nj}\Psi_{nj}(z_j)$$

3. Dissipative Systems and Stability

In this section we use the results of the dissipative systems theory[4],[5] to arrive at the criterion of stability for the system (1). Briefly stated the dissipative theory is used to derive an energy storage function, for a given dynamic system, which under certain restrictions can be used as a Lyapunov function. The stability criteria reported in[4] give conditions on the interconnection of a large scale system such that a weighted sum of the subsystem energy functions give a Lyapunov function for the overall system. Here we do not repeat the various proofs given in [4] and [5], instead we state only the claim as is relevant to our problem.

Claim 1

The system (1a) is (Q_i,S_i,R_i) dissipative for

$$Q_i = -\alpha_i,\quad s_i = [\alpha_i\beta_{i1} - \alpha_i\beta_{i2}],\quad R_i = \begin{bmatrix} \alpha_i r_{i1} & o \\ o & \alpha_i r_{i2} \end{bmatrix}$$

If there exists a positive definite matrix P_i matrices L_i and W_i satisfying the following equation (3). The storage function

$$\begin{aligned}
&\varnothing(\bar{x}_i) = \bar{x}_i^T P_i \bar{x}_i. \\
&P_i A_i + A_i^T P_i = C_i^T Q_i C_i - L_i L_i^T \\
&P_i B_i = C_i^T(A_i D_i + S_i^T) - L_i W_i \\
&R_i + S_i^T D_i + D_i^T S_i + D_i^T Q_i D_i = W_i^T W_i
\end{aligned} \tag{3}$$

Where subsystem (1a) for $i = 1,\dots,m$ has the system matrices $[A_i, B_i, C_i, D_i]$ and $\bar{x}_i^T = [x_i, x_n, z_i]$

Proof Refer [5].

Claim 2

If the function $f_k(\cdot)$ lies in the sector $[o,k_i]$, where $k_i>o$, then the subsystem

$$\begin{aligned}
\dot{\sigma}_k &= -\beta\sigma_k + u_k \\
y_k &= f_k(\sigma_k)
\end{aligned}$$

is $(-\alpha_i, \alpha_i\beta_i, o)$ dissipative, for any $\alpha_i > o$ and $\beta_i \geq \frac{k_i}{2\beta}$.

The storage function is $\varnothing(\sigma_k) = 2\alpha_i\beta_i \int_o^{\sigma_k} f_k(\sigma)d\sigma$

Proof Refer [5].

The Claims 1 and 2 give us the storage functions for each m linear subsystem (1a) and $m^2 + 2m$ nonlinear dynamic subsystem (1b) respectively. The overall system (1) can be obtained as a linear interconnection of

(i) m - linear subsystems (2a),
(ii) $m^2 + 2m$ nonlinear dynamic subsystems (1b)

let $y^T = [y_1,\dots,y_m,y_{m+1},\dots,y_{m^2+3m}]$
$u^T = [u_{11},u_{12},\dots,u_{m1},u_{m2}u_{m+1},\dots,u_{m^2+3m}]$

and $u = -Hy$ specify the linear interconnections to get the composite system (1). H is called the interconnection matrix.

We go through the following steps to check for stability of an interconnected system.

Step 1 Use Claim 1 to get (Q_i,S_i,R_i) for all the m linear subsystems (1a).

Step 2 Use Claim 2 to get (Q_i,S_i,R_i) for all the m^2+2m nonlinear dynamic systems (1b).

Step 3 Define

$$\begin{aligned}
Q &\triangleq \mathrm{diag}\,(Q_1,\dots,Q_{m^2+3m}) \\
S &\triangleq \mathrm{diag}\,(S_1,\dots,S_{m^2+3m}) \\
R &\triangleq \mathrm{diag}\,(R_1,\dots,R_{m^2+3m})
\end{aligned}$$

Step 4 Using the interconnection matrix H define the matrix $\hat{Q}$ as

$$\hat{Q} = SH + H^T S^T - H^T RH - Q$$

and check if $\hat{Q}$ is a positive definite matrix.

Step 5 If $\hat{Q}$ is a positive definite matrix then the system (1) is stable

and $\varnothing(\bar{x}) = \sum_{i=1}^{m^2+3m} \varnothing_i(\bar{x}_i)$

is a Lyapunov function.

If $\hat{Q}$ is not a positive definite function start again from Step 1 choosing different set of dissipativity parameters.

Note that if a subsystem is (Q_i,S_i,R_i) dissipative then it is also $(\alpha_i Q_i,\alpha_i S_i,\alpha_i R_i)$ dissipative for all $\alpha_i>o$.

We call α_i the scaling factors. There are no set guidelines to choose α_i's, but we choose them in such a way as to increase the chances of $\hat{Q}$ being positive definite. One possible choice is to select the scaling factors so as to cancel as many as possible off diagonal terms of $\hat{Q}$. In the next section we see how these results can be used to obtain stability regions for power systems.

4. Multimachine Power System

In this section we derive the Lyapunov function for a power system with transfer conductances and non-unifrom damping. We use the formulation of section 2 and the method of section 3

Using the standard nomenclature [6] we can represent the i^{th} machine of an n-machine system as

$$M_i\ddot{\delta}_i + d_i\dot{\delta}_i + \sum_{\substack{j=1\\ j\neq i}}^{n} E_iE_jY_{ij}[\sin(\delta_{ij}+\Theta_{ij})$$

$$-\sin(\delta_{ij}+\Theta_{ij})] = o$$

$$\text{for} \quad i = 1,2,..,n. \tag{4}$$

Define the state variables as

$$x_i \triangleq \dot{\delta}_i \qquad i = 1,..,n$$
$$z_i \triangleq \delta_{in} - \delta_{in}^{o} \quad i = 1,..,m$$
$$m = n-1$$
$$\delta_{in} \triangleq \delta_i - \delta_n$$

The state space representation of system (4) using the above defined state variables is

$$\dot{x}_i = -\lambda_i x_i + u_i \quad i = 1,..,m$$
$$\dot{x}_n = -\lambda_n x_n + u_n$$
$$\dot{z}_i = x_i - x_n$$

where

$$u_i = -b_{in}[\sin(z_i+\delta_{in}+\Theta_{in}) - \sin(\delta_{in}+\Theta_{in})]$$
$$-\sum_{\substack{j=1\\ j\neq i}}^{m} b_{ij}[\sin(z_i - z_j + \delta_{ij} + D_{ij}) - \sin(\delta_{ij}+\theta_{ij})]$$
$$u_n = -\sum_{j=1}^{m} b_{nj}[\sin(z_j - (\delta_{nj}+\Theta_{nj})) + \sin(\delta_{nj}+\Theta_{nj})]$$

after a simple algebraic manipulation the above set of equations can be written as

$$\dot{x}_i = -\lambda_i x_i - \mu_{i1} z_i + u_{i1} \quad i = 1,2,\ldots m$$
$$\dot{x}_n = -\lambda_n x_n + \mu_{i2} z_i + u_{i2}$$
$$\dot{z}_i = x_i - x_n \tag{5}$$

where

$$u_{i2} = b_{in}[\sin(z_i + \delta_{in} + \Theta_{in}) - \sin(\delta_{in}+\Theta_{in}) - \varepsilon_{i1} z_i]$$
$$-\sum_{\substack{j=1\\ j\neq i}}^{m} b_{ij}[\sin(z_i - z_j + \delta_{ij} + \Theta_{ij}) - \sin(\delta_{ij}+\Theta_{ij})]$$
$$u_{i2} = b_{ni}[\sin(z_i - (\delta_{ni}+\Theta_{ni})) + \sin(\delta_{ni}+\Theta_{ni}) - \varepsilon_{i2} z_i]$$
$$+\sum_{\substack{j=1\\ j\neq i}}^{m} b_{nj}[\sin(z_i - (\delta_{nj}+\Theta_{nj})) + \sin(\delta_{nj}+\Theta_{nj})]$$

where

$$o \le \varepsilon_{i1} \le 1$$
$$o \le \varepsilon_{i2} \le 1$$

with

$$\mu_{i1} = b_{in}\varepsilon_{i1}$$

and

$$\mu_{i2} = b_{ni}\varepsilon_{i2}$$

We can see that system (5) is a special case of system (2), where $\Psi ij(\cdot)$ are the various sinusoidal nonlinearities. This being the case, we now give a step-by-step procedure to construct the Lyapunov function of system (5).

Step 1 The system matrices for the i^{th} linear subsystem are

$i = 1,...,m$

$$A_i = \begin{bmatrix} -\lambda_i & o & -\mu_{i1} \\ o & -\lambda_n & \mu_{i2} \\ 1 & -1 & o \end{bmatrix}; \beta_i = \begin{bmatrix} 1 & 0 \\ o & 1 \\ 0 & 0 \end{bmatrix}$$
$$C_i = [1 \;\; -1 \;\; \beta], D_i = o$$

for $\quad o \le \varepsilon_{i1}, \varepsilon_{i2} < 1$

$$a_i \triangleq \lambda_i + \lambda_n$$
$$b_i \triangleq \mu_{i1} + \mu_{i2} + \lambda_i\lambda_n$$

let $\quad c_i \triangleq \varepsilon_{i2} b_{ni}\lambda_i + \varepsilon_{i2} b_{in}\lambda_n$

then choose ε_{i1}, and ε_{i2} such that

$$a_i^2 - 2b_i > o$$

$$(a_i^2 - 2b_i)(b_i^2 - 2a_i c_i) - c_i^2 > o$$

initial choice can be $\varepsilon_{i1} = \varepsilon_{i2} = 0.9$

$$\beta = \sum_{i=1}^{n} \lambda_i / n$$

$$\beta_{i1} > \max\left[\frac{1}{2(\lambda_i - \beta)}, \frac{\beta^2 + \lambda_n^2}{2(\lambda_n b_i + \beta b_i - a_i\beta\lambda_i - C_i)}, \frac{\beta\lambda_n}{2C_i}\right]$$

$$\beta_{i2} > \max\left[\frac{1}{2(\lambda_n - \beta)}, \frac{\beta^2 + \lambda_i^2}{2(\lambda_i b_i + \beta b_i - a_i\beta\lambda_i - C_i)}, \frac{\beta\lambda_n}{2C_i}\right]$$

such that $\beta_{i1}\lambda_n = \beta_{i2}\lambda_i$

$$r_{i1}, r_{i2} = 0.1$$

Now solve for the symmetric positive definite matrix P_i the matrix equation (3). If the matrix equation (3) does not have a solution for the chosen value of (r_{i1}, r_{i2}), we have two options

(a) increase the value of (r_{i1}, r_{i2})

(b) increase the value of $(\varepsilon_{i1}, \varepsilon_{i2})$

and then solve (3). In the case where equation (3) has a solution, the i^{th} subsystem is

$$\left(-\alpha_i, [\alpha_i\beta_{i1} - \alpha_i\beta_{i2}], \begin{bmatrix} \alpha_i r_{i1} & o \\ o & \alpha_i r_{i2} \end{bmatrix}\right)$$

with the storage function

$$\varnothing_i(\tilde{x}_i) = \tilde{x}_i^T P_i \tilde{x}_i.$$

Note that the large values of $r_{i1}, r_{i2}, \varepsilon_{i1}, \varepsilon_{i2}$ ensure the satisfaction of conditions put by Claim 1. At the same time large values of r_{i2} and r_{i2} reduce the chance of $\hat{Q}$ being positive definite and the larger $\varepsilon_{i1}, \varepsilon_{i2}$ are the more conservative the stability region is, so the idea is to keep these parameters as close to zero as possible.

<u>Step 2</u> Choose β_i, $i = m+1, \ldots, m^2+2m$

$$\beta_i = \frac{k_{i-m}}{2\beta}$$

where the $f_{i-m}(\cdot)$ is in the sector (o, k_{i-m})

<u>Step 3</u> Scaling factors

Choose
$$\alpha_i \beta_{i1} = 1 \qquad i = 1, \ldots, m$$
$$\alpha_i \beta_{i1} b_{ij} = \alpha_{mi+j} \beta_{mi+j}, \qquad j = 1, \ldots, m, j \neq i$$
$$\alpha_i \beta_{i1} b_{in} = \alpha_{mi+i} \beta_{mi+i}$$
$$\alpha_i \beta_{i2} b_{ni} = \alpha_{m^2+m+i} \beta_{m^2+m+i}$$
$$\alpha_i \beta_{i1} b_{ni} = \alpha_{m^2+2m+i} \beta_{m^2+2m+i}$$

<u>Step 4</u> Form the matrix $\hat{Q} = SH + H^T S^T - H^T R - Q$ as discussed in section 3, check whether it is positive definite or note. If yes, then step 5 gives the Lyapunov function, if not choose a different set of ε_i's, possibly larger than the previous one and restart at step 1.

<u>Step 5</u> If $\hat{Q}$ is positive definite the Lyapunov function is given by

$$V = \sum_{i=1}^{m} \tilde{x}_i^T P_i \tilde{x}_i + \sum_{k=1}^{m} \sum_{\substack{j=1 \\ j \neq k}}^{m} 2\alpha_k \beta_{kl} b_{kj} \int_o^{\sigma_k} \psi_{kj}(\sigma) d\sigma$$
$$+ \sum_{k=1}^{m} 2\alpha_k \beta_{kl} b_{kn} \int_o^{\sigma_k} \psi_{kn}(\sigma) d\sigma + \sum_{k=1}^{m} 2\alpha_k \beta_{kl} b_{nk} \int_o^{\sigma_k} \psi_{nk_1}(\alpha) d\sigma$$
$$+ \sum_{k=1}^{m} 2\alpha_k \beta_{k2} b_{nk} \int_o^{\sigma_k} \psi_{nk}(\sigma) d\sigma$$

5. Numerical Example

Let us consider a 3-machine system with the following parameters in per unit

$M_1 = 0.1, M_2 = 0.1, M_3 = 4.0$
$E_1, = E_2 = E_3 = 1.0$

$Y_{12} = 0.05, Y_{13} = 2.0, Y_{23} = 2.0$

$\delta^\circ_{12} = 5^\circ, \delta^\circ_{13} = 2^\circ, \delta^\circ_{23} = -3^\circ$

$\theta_{12} = 4^\circ, \theta_{13} = 2^\circ, \theta_{23} = 1^\circ$
$\lambda_1 = 9.5, \lambda_2 = 10.0, \lambda_3 = 10.5$

with these values we have

$b_{12} = b_{21} = 0.5, b_{13} = b_{23} = 20.0, b_{31} = b_{32} = 0.5$

choose $\beta = 5.0$
$\varepsilon_1 = \varepsilon_2 = 0.5$

using the procedure discussed in section 4 we get

$\beta_{11} = 0.244, \beta_{12} = 0.270$
$r_{11} = 0.1, r_{12} = 2.1$
$\beta_{21} = 0.244\ \beta_{22} = 0.256$
$r_{21} = 0.1, r_{22} = 2.2$

$$\alpha_1 \beta_{11} = \alpha_2 \beta_{21} = 1.0 \Rightarrow \alpha_1 = \frac{1}{\beta_{11}}, \alpha_2 = \frac{1}{\beta_{21}}$$

$$P_1 = \begin{bmatrix} 0.1836 & -0.1697 & 1.2232 \\ -0.1697 & 0.1572 & -1.1203 \\ 1.2232 & 01.1203 & 8.4888 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 0.174 & -0.168 & 1.221 \\ -0.168 & 0.1619 & -1.169 \\ 1.221 & -1.169 & 8.995 \end{bmatrix}$$

With the given values of ε_i's the nonlinearities must lie in the sector

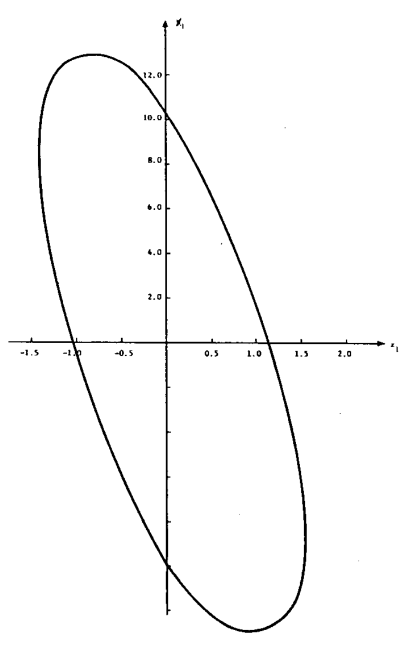
$-3.1763 \leq z_1 - z_2 \leq 2.8275$
$-1.895 \leq z_1, \leq 1.782$
$-1.838 \leq z_2 \leq 2.005$

The Lyapunov function V is found as reported in Step 5 of the previous section. The critical value of V is obtained by minimising it over the polygon formed by the above mentioned constraints. Numerical optimization gives $V_{cr} = 18.5187$ at

$x_1 = -51.75, x_2 = -42.23, x_3 = -43.1371$
$z_1 = 1.7820, z_2 = 0.1025$

The region of stability is given by $\{\tilde{x} : V(\tilde{x}) < 18.5187\}$

Figure 1 shows the x_1-z_1 cross section of the estimated region of stability.



<u>Fig. 1</u> Region of Stability

6. Conclusion

A systematic procedure is given to construct a Lyapunov function for general n-machine power system with transfer conductances. The n-machine system is decomposed into smaller subsystems and is first analysed at subsystem level then a composition is done to study the overall system stability. This simple method and the use of input-output stability concepts to choose suitable multiplier has helped in constructing a new Lyapunov function. A numerical example demonstrates the procedure. We obtain a region of stability using this approach while many previously reported methods could not produce any. A proper way to choose the scaling factors will help in getting sharper results.

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