Subject to the condition $R(\Gamma_T) < 1$, which implies that the matrix $(I - \Gamma_T)^{-1}$ is nonnegative, the following inequality is derived from (36):

$$\langle \|P_T\dot{y}\|_1\rangle \leqslant (I-\Gamma_T)^{-1}\langle \|P_T\dot{y}_L\|_1\rangle. \tag{37}$$

Condition 2 of the theorem ensures that inequality (37) holds true for all truncations $0 \leqslant T \leqslant \infty$ and thus bounds the total variation of the nonlinear response with respect to the total variation of the linearized system's response. Provided that the linearized system is stable, its response to bounded inputs with finite total variation (i.e., inputs which converge to a set-level, such as step inputs) exhibits finite total variation. Consequently $\langle \|\dot{y}_L\|_1\rangle$ is finite and from (37) $\langle \|\dot{y}\|_1\rangle$ is also finite. This proves the existence of nonoscillatory equilibria for the nonlinear system in response to bounded inputs with finite total variation. In order to complete the proof, we must show that any motion, which is caused by some initial perturbation from an equilibrium point, converges to the same equilibrium point.

Let y^* denote any nonoscillatory equilibrium point for the nonlinear system. As such, y^* satisfies the loop equation:

$$y^* = y_L^* - SE(y^*) \tag{38}$$

where y_L^* is the corresponding linearized system's equilibrium. Subtracting (38) from (22) gives:

$$y - y^* = y_L - y_L^* - S\{E(y) - E(y^*)\}. \tag{39}$$

Taking norms in $\mathfrak{L}_1$ and invoking the condition $\lim_{T\to\infty} R(\Gamma_T) < 1$, it is easily shown that:

$$\langle \|y - y^*\|_1\rangle \leqslant (I-\Gamma_\infty)^{-1}\langle \|y_L - y_L^*\|_1\rangle. \tag{40}$$

Since the linearized system is assumed stable, any motion $y_L(t)$ that is caused by some initial perturbation from its equilibrium y_L^* converges to y_L^*, and therefore $\langle \|y_L - y_L^*\|_1\rangle$ is finite. Inequality (40) ensures that the same argument applies to the corresponding nonlinear system's equilibrium point y^*. Consequently, the conditions of Theorem 2 imply the asymptotic stability of any given equilibrium state of the nonlinear system. A controllability and observability requirement can be imposed on Q so as to exclude the possbility of unstable system states that are unobservable at the output.

The practical importance of Theorem 2 lies in that it ensures the absence of limit cycling phenomena in the response of the nonlinear system to inputs that exhibit finite total variation (e.g., step inputs).

Concluding Remarks

The absolute stability criteria for multivariable nonlinear feedback systems that have been described in this paper, are easily verified conditions which involve the linearized system's "absolute gain," $\langle \|S\|_\infty\rangle$ and the static or incremental gain matrices $\langle \bar{E}\rangle$ or $\langle \hat{E}\rangle$ of the nonlinear error function $E(\cdot) = N(\cdot) - L$. The practical importance of the results lies in their dependence on the stability and the performance of the linearized system which often is the design objective in linearization-based design methods for nonlinear systems [3].

A direct comparison between the previous absolute stability criteria and well-known stability criteria such as the small gain theorem [5], [6], or frequency domain criteria such as Popov's criterion and the circle criteria [7], [8], would be rather unproductive as different stability criteria depend on different descriptions of the linear and nonlinear parts of the system. Furthermore, Theorem 2 is a stronger asymptotic stability criterion in that it ensures the asymptotic stability of any equilibrium state of the nonlinear system. However, a comparison between the stability criterion of Theorem 1 and the frequency domain stability criteria is possible on the basis that, like Theorem 1, the latter ensure the asymptotic stability of the system's zero equilibrium solution only; referred to as the "autonomous system." In this context, a disadvantage of Theorem 1 lies in the fact that the "absolute gain" matrix $\langle \|S\|_\infty\rangle$, which is involved in the matrix Λ_∞, is a rather conservative description of the linearized system's properties. In contrast, frequency domain absolute stability criteria employ the more descriptive frequency response of the linear system and therefore, it is reasonable to expect the latter to be less conservative than the conditions of Theorem 1.

Nevertheless, the practical advantages of Theorem 1, as well as Theorem 2, lie in the computational simplicity with which the asymptotic stability properties of multivariable nonlinear systems can be verified, and their direct applicability to the time domain analysis and design of nonlinear systems via linearization.

Acknowledgment

The authors express their thanks to Dr. D. H. Owens of the University of Sheffield, U.K., for his useful suggestions regarding the proofs of the theorems. This work forms part of a collaborative research program with the above colleague.

References

[1] D. H. Owens, "On the effect of nonlinearities in multivariable first order process control," in *IEE Conf. Publ.* 194, 1981, pp. 240-244.
[2] J. O. Gray and D. Valsamis, "Functional bounds on the time response of multivariable nonlinear systems," Centre for Instr. Automat., Univ. Salford, Salford, U.K., 1983.
[3] ——, "Design of multivariable nonlinear systems," presented at the IFAC Congress, Budapest, Hungary, 1984.
[4] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables.* New York: Academic, 1970.
[5] C. A. Desoer and M. Vidyasagar, *Feedback Systems: Input-Output Properties.* New York: Academic, 1975.
[6] J. M. Holtzman, *Nonlinear Systems Theory.* Englewood Cliffs, NJ: Prentice-Hall, 1970.
[7] J. L. Willems, *Stability Theory of Dynamical Systems.* New York: Nelson, 1970.
[8] H. H. Rosenbrock, "Multivariable circle theorems," in *Proc. IMA Conf. on Recent Math. Devel. in Contr.,* Bath, England, 1972, pp. 345-365.

Lyapunov Function for Power System with Transfer Conductances

HEMANSHU ROY POTA AND PETER J. MOYLAN

Abstract—**A method is given for deriving a Lyapunov function for a two-machine power system with nonzero transfer conductances. The assumptions required are only mildly restrictive and are likely to be satisfied for practical values of machine parameters. The method shows promise for extension to systems of more than two machines.**

I. Introduction

This note gives a systematic procedure for deriving a quadratic Lyapunov function for a two-machine system with transfer conductance. The approach is based on the stability results given in [11], for large-scale interconnected systems, using dissipative systems theory [10], [12]. This derivation is seen as one of the building blocks in a systematic procedure to derive a Lyapunov function, using a decomposition approach, for a general multimachine system including transfer conductances.

Researchers have been looking into direct methods of power system transient stability analysis for the last twenty years [16]. Direct methods not only save a lot in computer time, but also provide a deeper insight (compared to the step-by-step method) into transient stability behavior [17].

Manuscript received January 3, 1984; revised January 30, 1984.
The authors are with the Department of Electrical Engineering, University of Newcastle, New South Wales, 2308, Australia.

Willems [1] has given a systematic procedure to derive all known Lyapunov functions for a multimachine system, excluding transfer conductances. Systems with transfer conductance can be thought as one step higher in complexity. It is tempting to try to generalize all these Lyapunov functions for system with transfer conductance. Unfortunately, all the attempts to derive a Lyapunov function in the manner of Aylett, with path independent integral terms, have failed [9], although there exist some approximate [8], [9] and numerical methods [7]. The reasons for this failure, which are well discussed in [3], [4], [9], do not rule out the possibility of finding a good quadratic Lyapunov function. In this note we show how to find such a function.

Jočic *et al.* [13] developed a general procedure to derive Lyapunov functions for a multimachine system using a decomposition-aggregation technique (transfer conductance included). Araki [15], using an approach based on [14], improved upon [13]. Both of these methods give valid Lyapunov functions only when the damping is uniform and large. The method in [13] appears not to work for any practical value of damping coefficient. The procedure given in this note will give valid Lyapunov functions for most practical values of the damping coefficient.

Section II of this note gives the system model and the chosen decomposition. Section III outlines a procedure to choose the dissipativeness parameters. Section IV establishes the Lyapunov function validity. Section V contains a numerical example.

II. System Model

The differential equation describing the motion of the ith machine in a multimachine power system is given by

$$M_i \frac{d^2\delta_i}{dt^2} + d_i \frac{d\delta_i}{dt} = P_{m_i} - P_{e_i} = \sum_{\substack{j=i \\ j\neq i}}^{n} E_iE_jY_{ij}[\sin(\delta_{ij}^0 + \theta_{ij})$$

$$-\sin(\delta_{ij}+\theta_{ij})] \quad i = 1, 2, \cdots, n.$$

Nomenclature is the same as in [9].

We decompose the two-machine system into three subsystems.

Subsystem 1:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\lambda_1 & 0 & 0 \\ 0 & -\lambda_2 & 0 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y_1 = [0 \quad 0 \quad 1] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The system transfer function is

$$G(s) = \begin{bmatrix} \dfrac{1}{s(s+\lambda_1)} & \dfrac{1}{s(s+\lambda_2)} \end{bmatrix}.$$

Subsystem 2: (Memoryless)

$$y_2 = \sin(u_3 + \delta_{12}^0 + \theta_{12}) - \sin(\delta_{12}^0 + \theta_{12}).$$

Subsystem 3: (Memoryless)

$$y_3 = \sin(u_4 - (\delta_{21}^0 + \theta_{12})) + \sin(\delta_{21}^0 + \theta_{12}).$$

The interconnection relation represented in matrix form is

$$u_1 \triangleq -b_{12}\{\sin(x_3 + \delta_{12}^0 + \theta_{12}) - \sin(\delta_{12}^0 + \theta_{12})\}$$

$$u_2 \triangleq b_{21}\{\sin(x_3 - (\delta_{21}^0 + \theta_{12})) + \sin(\delta_{21}^0 + \theta_{12})\}$$

$$b_{ij} = \frac{E_iE_jY_{ij}}{M_i}; \quad \lambda_i = \frac{d_i}{M_i}$$

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = -\begin{bmatrix} 0 & b_{12} & 0 \\ 0 & 0 & -b_{21} \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}.$$

III. Dissipativeness Parameters

Our approach to deriving a Lyapunov function is based on the stability theory of [10]–[12]. In particular, we use the main stability theorems of [11], which show how to generate a Lyapunov function provided that each subsystem has a property called (Q, S, R) dissipativeness. To apply those results, we must find for the ith subsystem, matrices Q_i, S_i, and R_i such that the subsystem is (Q_i, S_i, R_i) dissipative. The choice of matrices is not unique, and a certain amount of trial and error was needed before settling on the choice shown below.

Definition: A system is (Q, S, R) dissipative if

$$\int_0^T (y'(t)Qy(t) + 2y'(t)Su(t) + u'(t)Ru(t))dt \geq 0$$

whenever $x(0) = 0$, for all admissible inputs u and $\forall T \geq 0$.

Theorem 1: Subsystem 1 is

$$\left([0], \begin{bmatrix} \dfrac{\lambda_2}{\lambda_1}\alpha_1 & -\alpha_1 \end{bmatrix}, \begin{bmatrix} \alpha_1\beta_1 & 0 \\ 0 & \alpha_1\beta_2 \end{bmatrix} \right)$$

dissipative, provided

$$\beta_1 > \frac{2\lambda_2}{\lambda_1^3} \tag{1}$$

$$\left(\beta_1 - \frac{2\lambda_2}{\lambda_1^3}\right)\left(\beta_2 - \frac{2}{\lambda_2^2}\right) > \frac{(\lambda_1+\lambda_2)^2}{\lambda_1^4\lambda_2^2} \tag{2}$$

for any $\alpha_1 > 0$.

Proof: Theorems 11 and 12 of [18] are used.

$$M(jw) = G^*(jw)QG(jw) + G^*(jw)S + S^TG(jw) + R$$

$M(jw) \geq 0$, $\forall w$, for the above choice of parameters.

The system is stable, if the feedback law is,

$$u = -Ky, \text{ where } k = \begin{bmatrix} k_1 \\ 0 \end{bmatrix}, \ k_1 > 0 \text{ sufficiently small.}$$

Theorems 11 and 12 of [18] are satisfied, hence the proof.

Subsystems 2 and 3 are memoryless conic nonlinearities. For $\alpha_2, \alpha_3 > 0$, subsystem 2 is $(-\alpha_2, \alpha_2(a_1 + b_1)/2, -\alpha_2a_1b_1)$ dissipative provided the nonlinearity lies in the sector (a_1, b_1); subsystem 3 is $(-\alpha_3, \alpha_3(a_2 + b_2)/2, -\alpha_3a_2b_2)$ dissipative provided the nonlinearity lies in the sector (a_2, b_2). If we were interested in global stability, then the obvious choice of sector would be $(-1, 1)$, since the nonlinearities are sinusoidal. However, tighter sector bounds are acceptable when, as in the case of a power system, only local asymptotic stability is of interest. In the following section, we consider the nonlinearities to be in the sector $(2\epsilon, 1)$, for an arbitrarily small $\epsilon > 0$.

IV. Derivation of Lyapunov Function

Results in [11] are used to prove local asymptotic stability. The stability proof is not in itself important because we normally start with a locally stable system. As a side effect, though, the stability proof gives us a Lyapunov function [10], and it is this Lyapunov function which is of interest.

In this section, we give a step-by-step method of choosing the dissipativeness parameters such that the stability criterion of [10] is satisfied. The end result of the calculation is a matrix P such that $x'Px$ is a Lyapunov function.

Step 1: Check that

$$\frac{1}{b_{12}} - \left(\frac{\lambda_2}{\lambda_1^2}\right)^2 > 0 \tag{3}$$

and

$$\left(\frac{1}{b_{12}}-\frac{\lambda_2^2}{(\lambda_1^2)^2}\right)\left(\frac{1}{b_{21}}-\frac{1}{\lambda_2^2}\right)>\frac{(\lambda_1+\lambda_2)^2}{4\lambda_1^5\lambda_2} \tag{4}$$

(if not true, the procedure fails).

Choose small ϵ, δ_1, $\delta_2 > 0$ such that

$$\frac{\lambda_1}{(0.5+\epsilon)b_{12}\lambda_2} > \frac{2\lambda_2}{\lambda_1^3}+\delta_1$$

$$\left(\frac{\lambda_1}{(0.5+\epsilon)b_{12}\lambda_2}-\frac{2\lambda_2}{\lambda_1^3}-\delta_1\right)\left(\frac{2}{b_{21}}-\frac{2}{\lambda_2^2}-\delta_2\right) > \frac{(\lambda_1+\lambda_2)^2}{\lambda_1^4\lambda_2^2}.$$

Step 2: Choose $a_1 = a_2 = 2\epsilon$

$$b_1=b_2=1.0$$

$$\alpha_1=\frac{(0.5+\epsilon)\lambda_2}{b_{12}\lambda_1} \qquad \alpha_2=1.0 \qquad \alpha_3=\frac{\lambda_2 b_{21}}{\lambda_1 b_{12}}$$

$$\beta_1=\frac{\lambda_1}{b_{12}\lambda_2(0.5+\epsilon)}-\delta_1 \qquad \beta_2=\frac{2}{b_{21}}-\delta_2.$$

Step 3: Solve for the matrices P (symmetric positive definite), L, and W the following matrix equations:

$$PA+A^TP=-LL^T$$

$$PB=CS-LW$$

$$R=W^TW.$$

A, B, C being system matrices, S, R being dissipativeness parameters for the subsystem 1.

The following theorem states the main result of this paper.

Theorem 2: If conditions (3) and (4) hold and the parameters are chosen according to Steps 1–3, then $V(x) = x^TPx$ is a valid Lyapunov function establishing local asymptotic stability of the original system.

Remark: The proof is based on [11, Theorems 1 and 2]. Let

$$Q \triangleq \operatorname{diag}(Q_i),\quad S \triangleq \operatorname{diag}(S_i),\quad R \triangleq \operatorname{diag}(R_i),$$

and

$$\hat{Q} \triangleq SH+H^TS^T-H^TRH-Q$$

where H is the matrix of interconnection coefficients. Theorem 1 [11] states that the system is input–output stable if $\hat{Q}$ is positive definite. Theorem 2 [11] extends this result to internal stability provided the state space satisfied certain conditions. Our proof will show that the above choice of parameters satisfy the conditions of Theorems 1 and 2 [11].

Proof:

$$\hat{Q}=\begin{bmatrix}\alpha_2 a_1 b_1+\alpha_3 a_2 b_2 & 0 & 0\\ 0 & \alpha_2-\alpha_1\beta_1 b_{12}^2 & 0\\ 0 & 0 & \alpha_3-\alpha_1\beta_2 b_{21}^2\end{bmatrix}.$$

Sinusoidal nonlinearity is in the sector $(2\epsilon, 1)$ giving

$$\frac{a_1+b_1}{2}=\frac{a_2+b_2}{2}=(0.5+\epsilon).$$

For the choice of α_i's given in Step 2, $\hat{Q}$ is positive definite if β_i's satisfy the following condition:

$$\beta_1<\frac{\lambda_1}{(0.5+\epsilon)b_{12}\lambda_2} \quad \beta_2<\frac{2}{b_{21}}.$$

Then, if the conditions (3) and (4) are satisfied, subsystem 1 is

$$\left([0],\ \begin{bmatrix}\frac{\lambda_2}{\lambda_1}\alpha_1 & -\alpha_1\end{bmatrix},\ \begin{bmatrix}\alpha_1\beta_1 & 0\\ 0 & \alpha_1\beta_2\end{bmatrix}\right)$$

dissipative (Theorem 1).

The other two subsystems being memoryless do not contribute to the Lyapunov function. Theorem 2 [11] guarantees existence of a Lyapunov function. The procedure given in [10, Theorem 16] and Step 3 can be used to find an energy function for a linear dissipative system. $\nabla\nabla\nabla$

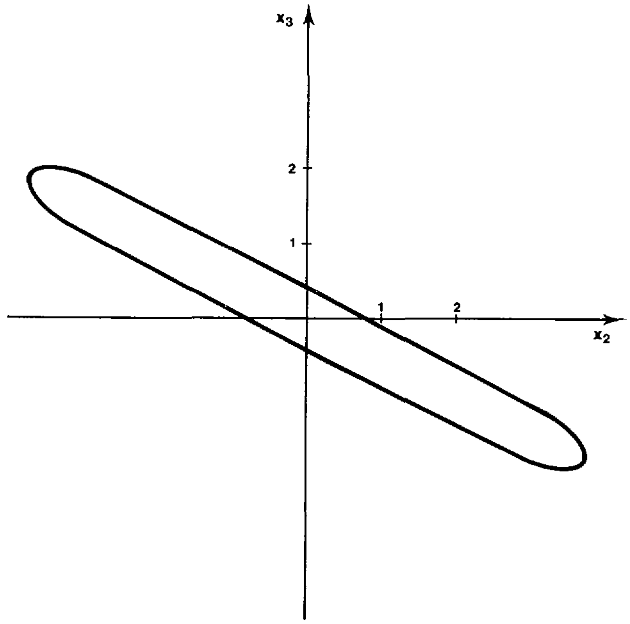

Fig. 1. Region of stability for the example, $x_1 = 0$.

Remark: This condition for the existence of Lyapunov function is more likely to be satisfied if the damping coefficients are large and the ratio of the damping coefficients is near unity. This shows that this condition is in line with the conjecture put forward by Willems [17]. This condition also depends upon interconnection coefficients. If b_{ij}'s are large the damping coefficients have to be large too. Intuitively we can see that strong interconnection will propagate more disturbance and for the synchronization to be maintained damping should be large.

This result can be repeatedly applied when dealing with a multimachine system. A common way to decompose a multimachine system is to take one machine in common with all the other machines. This gives $(n - 1)$ two-machine subsystems and $n(n - 1)$ nonlinearities. Subsystem analysis for $(n - 1)$ two-machine subsystems is the same as given by Theorem 2 and can be repeatedly applied to get dissipativeness parameters. Finding dissipativeness parameters for the $n(n - 1)$ nonlinearities is no problem at all. Unfortunately, the conditions on $\hat{Q}$ to be positive definite cannot be simplified (as for the two-machine case) and Sylvester's criterion has to be used. The role of α_i's and β_i's in manipulating $\hat{Q}$ is being studied and will be reported shortly.

V. Numerical Example

Machine parameters are

$$\lambda_1=1.0;\ \lambda_2=2.0;\ M_1=M_2=1.0;\ d_1=1.0;$$

$$d_2=2.0;\ b_{12}=b_{21}=0.1;\ \delta_{12}^0=30°;\ \theta_{12}=4°.$$

One set of dissipative parameters found following Steps 1–3 is

$$\left([0],\ [20\quad -10],\ \begin{bmatrix}200 & 0\\ 0 & 200\end{bmatrix}\right)$$

and the Lyapunov function is

$$V=4.54x_1^2+0.52x_2^2+2x_3^2-2.18x_1x_2+4x_2x_3-2x_2x_3$$

$$x_{3l}=-\pi-2\delta_{12}^0+2\theta_{12}$$

$$x_{3u}=\pi-2(\delta_{12}^0+\theta_{12}).$$

Now let

$$V_{cr} = \min_{i=l,\,u} [\min \ V(x) : x_3 = x_{3i}].$$

Numerical optimization gives $V_{cr} = 0.2784$ at $x_1 = 0.0834$, $x_2 = 3.934$, $x_3 = 1.955$. The region of stability is given by

$$4.54x_1^2 + 0.52x_2^2 + 2x_3^2 - 2.18x_1x_2 + 4x_1x_3 - 2x_2x_3 < 0.2784.$$

Fig. 1 shows x_2, x_3 cross section, $x_1 = 0.0$.

VI. CONCLUSION

A systematic procedure is given to construct a quadratic Lyapunov function for a two-machine system with transfer conductance. Dissipativity theory is used to study stability of a system decomposed into subsystems. The idea is to extend this procedure for a general multimachine system. If an n-machine system is decomposed as one machine common to all other machines (a commonly chosen decomposition [13], [15]), then the procedure given in this note is the same for all the subsystems; only $\hat{Q}$ calculation will depend on a particular system. Attempts are being made to extend this method, results will be reported later. Efforts are also being made to understand the role of the multiplier α_i's and as to what its value should be for a multimachine system. As such, dissipativity theory shows good promise of obtaining better stability results for large-scale systems.

REFERENCES

[1] J. L. Willems, "A partial stability approach to the problem of transient power system stability," *Int. J. Contr.*, vol. 19, no. 1, pp. 1–14, 1974.

[2] M. A. Pai and P. G. Murthy, "On Lyapunov functions for power systems with transfer conductances," *IEEE Trans. Automat. Contr.*, vol. AC-18, pp. 181–182, Apr. 1973.

[3] V. E. Henner, "Comments on 'Lyapunov functions for power systems with transfer conductances,' " *IEEE Trans. Automat. Contr.*, vol. AC-19, pp. 621–623, Oct. 1974.

[4] J. L. Willems, "Comments on 'A general Lyapunov function for multimachine power systems with transfer conductances,' " *Int. J. Contr.*, vol. 23, no. 1, pp. 147–148, 1976.

[5] J. B. Moore and B. D. O. Anderson, "A generalization of the Popov criterion," *J. Franklin Inst.*, vol. 285, pp. 488–492, June 1968.

[6] U. Gudaru, "A general Lyapunov function for multimachine power systems with transfer conductances," *Int. J. Contr.*, vol. 21, no. 2, pp. 333–343, 1975.

[7] K. Uemura, J. Matuski, J. Yamada, and T. Tsuji, "Approximation of an energy function in transient stability analysis of power systems," *Elec. Eng. Japan*, vol. 92, no. 6, 1972.

[8] S. D. Varwandkar and M. A. Pai, "On the inclusion of transfer conductances in Lyapunov functions for multimachine power systems," *IEEE Trans. Automat. Contr.*, vol. AC-22, no. 6, pp. 983–985, Dec. 1977.

[9] M. El Guindi and M. Mansour, "Transient stability of a power system by the Lyapunov method considering the transfer conductances," *IEEE Trans. Power Apparatus Syst.*, vol. PAS-101, pp. 1088–1094, May 1982.

[10] D. J. Hill and P. J. Moylan, "Dissipative dynamical systems: Basic input–output and state properties," *J. Franklin Inst.*, vol. 309, pp. 327–357, May 1980.

[11] P. J. Moylan and D. J. Hill, "Stability criteria for large scale systems," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 143–149, Apr. 1978.

[12] P. J. Moylan, "Circle criteria revisited," in *Proc. I. E. Australia, Second Conf. on Contr. Eng.*, Newcastle, Aug. 1982, pp. 56–58.

[13] L. B. Jočic, M. Ribbens-Pavella, and D. D. Siljak, "Multimachine power systems: Stability, decomposition and aggregation," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 325–332, Apr. 1978.

[14] M. Araki, "Stability of large-scale nonlinear systems—Quadratic order theory of composite-system method using M-matrices," *IEEE Trans. Automat. Contr.*, vol. AC-23, pp. 129–142, Apr. 1978.

[15] M. Araki, M. Saeki, and B. Kondo, "Application of a new stability criterion of composite systems to multimachine power systems," *IEEE Trans. Automat. Contr.*, vol. AC-25, pp. 480–483, June 1980.

[16] M. A. Pai, *Power System Stability—Analysis by the Direct Method of Lyapunov*. Amsterdam, The Netherlands: North-Holland, 1981.

[17] J. L. Willems, "Optimum Lyapunov functions and stability regions of multimachine power systems," *Proc. IEE*, vol. 117, pp. 573–578, Mar. 1970.

[18] D. J. Hill and P. J. Moylan, "General instability results for interconnected systems," *SIAM J. Contr. Optimiz.*, vol. 21, pp. 256–279, Mar. 1983.

Criteria for Asymptotic Stability of Linear Time-Delay Systems

TAKEHIRO MORI

Abstract—**Several sufficient conditions which guarantee stability of linear time-delay systems are derived. Each of these results is expressed by a succinct scalar inequality and corresponds to a certain extent to the tradeoff between simplicity and sharpness.**

I. INTRODUCTION

Stability analysis of time-delay systems has been one of the main concerns of the researchers who would like to inspect the properties of such systems. In general, the introduction of time-delay factors makes the analysis much more complicated, and convenient methods to check stability have long been sought. In the existing stability criteria of the systems, mainly two ways of approach have been adopted. Namely, one direction is to contrive the stability condition which does not include information on the delay and the other is the methods which take it into account. The former case is often called the delay-independent criteria and generally provides nice algebraic conditions [1]–[3]. However, abandonment of information on the delay necessarily causes conservativeness of the criteria especially when the delay is comparatively small.

In this paper, several criteria for asymptotic stability of linear time-delay systems of the form $\dot{x}(t) = Ax(t) + Bx(t - \tau)$ are derived. Most of them are delay-dependent criteria and are expressed by simple inequalities. The main results of this paper are shown in the next section, where a basic proposition is first provided and thereby several stability criteria are derived. As an application, two examples are worked in Section III and some concluding remarks are given in Section IV. Before beginning, we define some notations. Let $\mu(X)$ be the matrix measure for $X \in C^{n\times n}$ derived from some matrix norm $\|X\|$. $\lambda_i(X)$ denotes the eigenvalues of X and Re $\lambda_i(X)$ and Im $\lambda_i(X)$ are the real and imaginary parts, respectively. The imaginary unit is represented by j, i.e., $j^2 \triangleq -1$ and I denotes the unit matrix as usual.

II. MAIN RESULTS

Let us investigate the stability properties of systems described by linear differential difference equation

$$\dot{x}(t) = Ax(t) + Bx(t-\tau) \tag{1}$$

where $A, B \in R^{n\times n}$, $x(t) \in R^n$, $\tau > 0$. The fundamental result which gives a condition for asymptotic stability of the system is summarized in the following theorem.

Theorem: The system (1) is asymptotically stable (AS), if

$$\begin{cases} \mu(A) + \max\limits_{y\in\Delta} \mu(Be^{-\tau yj}) < 0, & \text{for } \max\limits_{y\in\Delta} \mu(Be^{-\tau yj}) \geqslant -1/\tau \\ 1 + \tau \cdot \max\limits_{y\in\Delta} \mu(Be^{-\tau yj}) \cdot e^{1-\tau\mu(A)} < 0, & \text{for } \max\limits_{y\in\Delta} \mu(Be^{-\tau yj}) < -1/\tau \end{cases} \tag{2}$$

holds. Here Δ denotes the range of the values taken by the solution y of

$$y = \text{Im } \lambda_i(A + Be^{-\tau yj} \cdot e^{-\text{Re } \tau s}), \ \text{Re } s \geqslant 0, \ s \in C, \ \tau > 0. \tag{3}$$

To prove this theorem, we need some preliminary results shown below.

Lemma 1: The necessary and sufficient condition for the transcendental inequality

$$x \leqslant \alpha + \beta e^{-x}, \qquad \alpha, \ \beta \in R \tag{4}$$

Manuscript received March 12, 1984; revised May 7, 1984. This work was supported by the Alexander von Humboldt Foundation.

The author is with the Institut für Regelungstechnik, Universität Hannover, Hannover, Germany, on leave from the Automation Research Laboratory, Kyoto University, Kyoto, Japan.

0018-9286/85/0200-0158$01.00