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On the Stability and Well-Posedness of Interconnected Nonlinear Dynamical Systems

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I. INTRODUCTION

IN THE PAST and current research on stability problems there is a dichotomy between input–output and state-space approaches. To gain the full benefit of the two approaches it is desirable to have results showing when input–output stability implies Lyapunov stability (for some results in the opposite direction see [1], [2]). The best-known result along these lines is due to Willems [3], who shows that, with certain reachability and observability assumptions, an input–output stable system is globally asymptotically stable in the sense of Lyapunov. While this is a powerful result, it has two features that make it worthwhile to study the problem further: 1) in the case of the standard finite-dimensional linear time-invariant systems, Willems' result states the following: if the system is controllable, observable, and L_2 stable, then it is also globally asymptotically stable. However, in such a case it is well known that the above statement is true with controllability and observability replaced by the weaker requirements of stabilizability and detectability, respectively. Thus it is worth investigating whether a similar weakening is possible in the case of nonlinear systems as well. 2) The uniform observability property defined in [3] has the disadvantage that it is not necessarily preserved under feedback. Hence, given a large-scale interconnected system, one cannot verify uniform observability at the subsystem level. Thus it is desirable to define suitable properties for nonlinear systems that *are* preserved under arbitrary interconnection.

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In extending our results from single systems to interconnected systems, we require that the Cartesian product of the subsystem state spaces qualifies as a state space for the overall system. In this paper we present some sufficient conditions for this to hold. These conditions are the state space analogs of the input–output results in [4], and reduce to the classical "no delay-free loops" condition in the case of digital filters.

II. Notation and Definitions

Consider a system with state space X, input spaces $\mathcal{U}_e$ and $\mathcal{U}\subseteq\mathcal{U}_e$, output spaces $\mathcal{Y}_e$ and $\mathcal{Y}\subseteq\mathcal{Y}_e$, and some distinguished "initial state" set $\Omega\subseteq X$. The state space X is a normed space. The set Ω might contain only a single point such as the origin, or several equilibrium points, or even (the set of states comprising) a limit cycle. Elements of $\mathcal{U}$, $\mathcal{U}_e$ are functions of time, mapping R into U (the set of input values), and we assume that $P_T\mathcal{U}_e\subseteq\mathcal{U}\ \forall T\in R$, where P_T is the causal truncation operator defined by

$$(P_Tf)(t)=\begin{cases} f(t), & t\leq T\\ 0, & t>T.\end{cases} \tag{1}$$

Similarly, $\mathcal{Y}$ and $\mathcal{Y}_e$ consist of functions mapping R and Y, and we assume that $P_T\mathcal{Y}_e\subseteq\mathcal{Y}\ \forall T\in R$. The input–output relationship is given by

$$y=G(x_0,t_0)u \tag{2}$$

where $x_0\in\Omega$ and $t_0\in R$ denote the initial state and initial time, respectively, and $G(x_0,t_0)$ maps $\mathcal{U}_e$ into $\mathcal{Y}_e$ for all $x_0\in\Omega$ and all $t_0\in R$.

Typically, $X=R^n$ in the case of a finite-dimensional system, and X is a separable Hilbert space in the case of an infinite-dimensional system. For the input and output spaces typical examples are $U=R^k$, $\mathcal{U}=L_p^k$, and $\mathcal{U}_e=L_{pe}^k$ for some integer k and some $p\in[1,\infty]$, and $\mathcal{Y}=L_q^l$ for some integer l and some $q\in[1,\infty]$. The representation (2) takes into account the possibility that the input–output map depends on the initial state and time.

Definition 1:

$$\mathcal{K}(G(x_0,t_0))\triangleq\{u\in\mathcal{U}: G(x_0,t_0)u\in\mathcal{Y}\}. \tag{3}$$

Thus $\mathcal{K}(G(x_0,t_0))$ can be thought of as the set of all "stabilizing inputs" to the mapping $G(x_0,t_0)$. In this paper, for the most part we never explicitly compute $\mathcal{K}(G(x_0,t_0))$; however, if $G(x_0,t_0)$ represents a linear time-invariant system, then under certain conditions one can calculate $\mathcal{K}(G(x_0,t_0))$ (see [5] for details).

Definition 2: The system (2) is *input–output stable* (or *I/O stable* for short) if

$$\mathcal{K}(G(x_0,t_0))=\mathcal{U},\qquad \forall x_0\in\Omega,\ \forall t_0\in R. \tag{4}$$

Equivalently, G is I/O stable if, for all $x_0\in\Omega$ and $t\in R, G(x_0,t_0)$ maps $\mathcal{U}$ into $\mathcal{Y}$. Conversely, G is I/O unstable if there exist $x_0\in\Omega$, $t_0\in R$ and $u\in\mathcal{U}$ such that $G(x_0,t_0)u\in\mathcal{Y}_e\backslash\mathcal{Y}$.

To keep the exposition simple we assume that the input $u(.)$ and output $y(.)$ of the system (2) are related by

$$\dot{x}(t)=f(x(t),u(t),t),\qquad \forall t\geq t_0,\ x(t_0)=x_0 \tag{5a}$$

$$y(t)=h(x(t),u(t),t) \tag{5b}$$

where the functions f and h are such that, for all $x\in\Omega$, $t_0\in R$ and $u\in\mathcal{U}_e$, (5) has a unique solution consistent with (2). We use $\phi(x_0,t,t_0,u)$ to denote the value of this solution at time t.

We now present the definitions of reachability and "detectability" for nonlinear systems.

Definition 1: A state $x_1\in X$ is *reachable* $t_1\in R$ from Ω if there exists some finite $t^*\leq t_1$, such that for all $t_0\leq t^*$ there exist $x_0\in\Omega$ and $u\in\mathcal{U}_e$ such that $\phi(x_0,t_1,t_0,u)=x_1$. A set $X_1\subseteq X$ is reachable at t_1 from Ω if every $x\in X_1$ is reachable at t_1 from Ω.

Definition 2: The system (2) is *conditionally state-bounded* (CSB) at $t_0\in R$ if $x_0\in\Omega$ and $u\in\mathcal{K}(G(x_0,t_0))$ together imply that there exists a constant $M(u)$ such that $\|\phi(x_0,t,t_0,u)\|\leq M(u)\ \forall t\geq t_0$.

Definition 3: The system (2) is *conditionally state-asymptotic* (CSA) at $t_0\in R$ if $x_0\in\Omega$ and $u\in\mathcal{K}(G(x_0,t_0))$ together imply that $\lim d[\phi(x_0,t,t_0,u),\Omega]=0$, where $d(z,\Omega)=\inf_{x\in\Omega}\|x-z\|$ is the distance between z and Ω.

The reachability definition requires that, *starting at any time t_0 prior to t^**, we can go from a state in Ω at time t_0 to every state in X_1 at time t_1 (See also Lemma 2 in Section IV). CSB requires that if we start at $x_0\in\Omega$ and apply a stabilizing control $u\in\mathcal{K}(G(x_0,t_0))$, then the resulting state trajectory is bounded. CSA requires that, under the same conditions, the resulting state trajectory approaches Ω as $t\to\infty$. Clearly, if Ω is a bounded set, CSA implies CSB.

Both CSB and CSA are forms of detectability, though this may not be evident at first glance. Typically, we study the property CSB when $\mathcal{U}=L_\infty^k$ and $\mathcal{Y}=L_\infty^l$. Thus CSB is a formalization of the following concept: if the input is bounded and the output is bounded, then the state must be bounded. In the same way we typically study the property CSA when $\mathcal{U}=L_p^k$ and $\mathcal{Y}=L_p^l$ for some $p<\infty$. Thus CSA is a formalization of the following concept: if the input goes to zero and the output goes to zero, then the state approaches the "equilibrium set" Ω. In particular, if $u(.)\equiv 0$, we don't insist that $x(.)\equiv 0$; rather we only require that $x(.)$ is bounded (CSB) or approaches Ω(CSA). In other words, the part of the state that we cannot measure is well behaved.

III. Stability Results

A CSB or CSA system is one for which instability in the state is, in effect, reflected in the output. Thus we would expect that, given either of these properties, input–output stability would imply internal stability. The precise results are contained in the next two theorems.

Theorem 1: Suppose that $G(\Omega,R)$ is I/O stable, that $X_1\subseteq X$ is reachable from Ω at time t_1, and that system (2) is CSB at t_0, $\forall t_0\leq t_1$. Then $\phi(x_1,t,t_1,0)$ is bounded as a function of t, $\forall x_1\in X_1$.

In effect the theorem states that if a system is I/O stable and CSB, then all zero-input trajectories starting

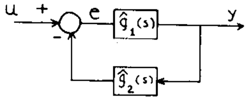

Fig. 1.

from reachable initial states are bounded. Roughly speaking, I/O stability, plus reachability, plus CSB, imply Lagrange stability.

Proof: Given $x_1 \in X_1$, select $t_0 < t_1$, $x_0 \in \Omega$, and $u \in \mathcal{U}_e$ such that $\phi(x_0, t_1, t_0, u) = x_1$. (This is always possible because X_1 is reachable at t_1 from Ω.) Since $u \in \mathcal{U}_e$ we have $P_{t_1}u \in \mathcal{U}$. Also, by I/O stability we have $\mathcal{K}(G(x_0, t_0)) = \mathcal{U}$, so that $P_{t_1}u \in \mathcal{K}(G(x_0, t_0))$. Finally, by CSB at t_0 we have $\phi(x_0, t, t_0, P_{t_1}u)(=\phi(x_1, t, t_1, 0)$ whatever $t \geq t_1)$ is bounded as a function of t. ■

In the preceding theorem the assumption that the system (2) is CSB at all $t_0 \leq t_1$ can be weakened, but at the expense of a more cumbersome statement.

Theorem 2: Suppose that $G(\Omega, R)$ is I/O stable, that $X_1 \subseteq X$ is reachable from Ω at time t_1, and that (2) is CSA at t_0 for all $t_0 \leq t_1$. Then $\phi(x_1, t, t_1, 0) \rightarrow \Omega$ as $t \rightarrow \infty$, $\forall x_1 \in X_1$.

Proof: Similar to that of Theorem 1. ■

As a consequence of these results the problem of checking internal stability can be replaced by the (often simpler) problems of checking reachability, I/O stability, and CSB or CSA. For small systems this may be only a minor gain. The real advantage comes when dealing with interconnected systems because both reachability and CSB (CSA) can be checked at the subsystem level, leaving only I/O stability, which can often be established by studying the properties of the various subsystems and interconnections.

Consider the interconnected system

$$u_i = u_{\text{ext}i} - \sum_{j=1}^{m} H_{ij} y_j$$

$$y_i = G_i(x_{0i}, t_0) u_i \qquad i = 1, \cdots m \tag{6}$$

where each G_i is a dynamical system with input, output, and state spaces $\mathcal{U}_{ei}$, $\mathcal{Y}_{ei}$, X_i, Ω_i, respectively; $u_{\text{ext}i} \in \mathcal{U}_{ei}$ $\forall i$; and H_{ij}: $\mathcal{Y}_{ej} \rightarrow \mathcal{U}_{ei}$ is a memoryless operator (i.e., $P_T H_{ij} P_T = P_T H_{ij}$, $Q_T H_{ij} Q_T = Q_T H_{ij}$) $\forall i, j$. We take $\mathcal{U}_e = \mathcal{U}_{ei} \times \cdots \times \mathcal{U}_{em}$ and $\mathcal{Y}_e = \mathcal{Y}_{el} \times \cdots \times \mathcal{Y}_{em}$ to be the input and output spaces for the overall system. We also make the important

Assumption (A1): The overall system (6) is a dynamical system with state space $X = X_1 \times \cdots \times X_m$ and initial state set $\Omega = \Omega_1 \times \cdots \times \Omega_m$.

This assumption does not hold for several innocent looking systems, such as the one shown in Fig. 1, with $\hat{g}_1(s) = s/(s+1)$ and $\hat{g}_2(s) = s/(s+2)$. Section V contains some simple sufficient conditions for the above assumption to hold.

With this setup we can state some results concerning reachability and CSB (CSA) of the *overall* system in terms of the subsystem properties.

Theorem 3: Suppose (A1) holds, that $\mathcal{U}_e$ is closed under addition, and that $X_{i1} \subseteq X_i$ is reachable from Ω_i at time t_1 for each subsystem G_i. Then $X_1 = X_{11} \times \cdots \times X_{m1}$ is reachable from Ω at time t_1 for the system (6).

Proof: Given $x_{i1} \in X_{i1}$, select $t_{0i} \leq t_1$, $z_{0i} \in \Omega_i$, and $u_i \in \mathcal{U}_{ei}$ such that $\phi_i(z_{0i}, t_i, t_{0i}, u_i) = x_{i1}$, for each i. Let $t_0 = \min t_{0i}$; then, by the definition of reachability, there exists $x_{0i} \in \Omega_i$ and $u_i \in \mathcal{U}_{ei}$ such that $\phi(x_{01}, t_i, t_0, u_i) = x_{1i}, \forall i$. If we now apply the external control defined by

$$u_{\text{ext}i} \triangleq u_i + \sum_{j=1}^{m} H_{ij} G_j(x_{0j}, t_0) u_j \tag{7}$$

it is easy to see that the state at time t_1 of the system (6) will be $(x_{11}, \cdots, x_{m1})$. ■

Theorem 4: If each subsystem G_i is CSB (CSA) at t_0, each H_{ij} maps $\mathcal{Y}_j$ into $\mathcal{U}_j$, and each $\mathcal{U}_i$ is closed under addition, then the system (6) is CAB (CSA) at t_0.

Proof: Let $G(x_0, t_0)$ denote the relation between u_{ext} and y and suppose $u_{\text{ext}} \in \mathcal{K}(G(x_0, t_0))$; then for each i we have $u_{\text{ext}\,i} \in \mathcal{U}_i$ and $y_i \in \mathcal{Y}_i$. Since H_{ij} maps $\mathcal{Y}_j$ into $\mathcal{U}_j$ for all i, j, this implies that $H_{ij} y_j \in \mathcal{U}_i$ $\forall i, j$, so that $u_i \in \mathcal{U}_i \forall i$. Next $u_i \in \mathcal{U}_i$, $y_i \in \mathcal{Y}_i$ implies that $u_i \in \mathcal{K}(G_i(x_{i0}, t_0))$. Since each subsystem is CSB (CSA) at t_0, we now have that $\phi(x_{i0}, t, t_0, u_i)$ is bounded $\forall i$ (approaches Ω_i for all i). Hence the system (6) is CSB (CSA) at t_0. ■

Suppose now that the large-scale system (6) is known to be I/O stable (though the subsystems might not be). Theorems 1 and 2 provide a means of checking whether the system is also internally stable; most important, Theorems 3 and 4 show that the hypotheses of Theorems 1 and 2 can be checked at the subsystem level.

IV. Testing for Reachability, CSA and CSB

In this section we discuss various ways of testing whether a given system has the properties of reachability, CSA and CSB. Specifically, if the initial state set Ω is invariant in a sense to be made precise below, we show that the definition of reachability can be replaced by an equivalent property that is easier to check. Further, we relate CSA and CSB to detectability in the case of liner time-invariant finite-dimensional systems. In view of Theorem 4 these relationships also apply in the case of Lur'e type feedback systems, as well as an arbitrary interconnection of linear time-invariant subsystems via memoryless interconnections.

Definition 4: Given the system (5) the set Ω is said to be *negatively invariant* if the following is true: For every $t_1 \in R$, $t_0 \leq t_i$, and $x_1 \in \Omega$, there is an $x_0 \in \Omega$ such that $\phi(x_0, t_i, t_0, 0) = x_1$.

In case Ω consists of a set of equilibrium points, the above condition is satisfied with $x_0 = x_1$. In case (5) represents a time-invariant system and Ω consists of a limit cycle of period T, the above condition is satisfied with $x_0 = \phi(x_1, \bar{t}, t_1, 0)$, where

$$\bar{t} = t - (t_1 - t_0) \bmod T. \tag{8}$$

This is the motivation for calling Ω negatively invariant.

Lemma 2: Suppose Ω is negatively invariant with respect to ϕ. Then a state $x_1 \in X$ is reachable from Ω at time t_i if and only if there exists some finite $t^* \leq t_1$, and $x_0 \in \mathcal{U}$ and $u \in \mathcal{U}_e$ such that $\phi(x_0, t_1, t^*, u) = x_1$.

Proof: Obvious.

A comparison of Lemma 2 with Definition 1 clearly brings out the simplification involved in case Ω is negatively invariant.

We turn now to CSA and CSB and focus our attention on linear systems of the form

$$\dot{x}(t) = Ax(t) + Bu(t) \qquad y(t) = Cx(t) \tag{9}$$

where $x = R^n$, $U = R^k$, $Y = R^l$.

Lemma 3: Suppose the system (9) is detectable and that $p < \infty (p = \infty)$. Then (9) is CSA (CSB) with $\Omega = \{0\}, t_0 = 0$.

The proof is simple and is, therefore, omitted.

We conclude this section by working out an example of a CSB system whose input and output spaces are L_2 (and not L_∞ as in Lemma 3).

Example: Consider the forced van der Pol oscillator

$$\dot{x}_1 - x_2 \quad \dot{x}_2 = x_1 + \mu(1 - x_1^2)x_2 + u; \quad y = x_2 \tag{10}$$

with $\mathcal{U} = \mathcal{Y} = L_2$. Let $v(x) = x_1^2 + x_2^2)/2$. Then

$$\dot{v} = \mu(1 - x_1^2)x_2^2 + x_2 u$$
$$\leq \mu x_2^2 + x_2 u \leq \mu\left(x_2 + \frac{1}{2\mu}u\right)^2 \tag{11}$$

so

$$v(x(T) \leq \mu \int_0^T \left[x_2(t) + \frac{1}{2\mu}u(t)\right]^2 dt$$
$$\leq \mu \|y(.)\|^2 + \frac{1}{2\mu}\|u(.)\|^2 \qquad \text{if } y \in L_2 \ u \in L_2. \tag{12}$$

Hence, this system is CSB with $\Omega = \{0\}, t_0 = 0$.

V. Well-Posedness

Recall now that we are studying an interconnection of several smooth dynamical systems, described by

$$\dot{x}_i(t) = f_i(x_i(t), u_i(t), t) \tag{13a}$$
$$y_i(t) = h_i(x_i(t), u_i(t), t) \tag{13b}$$

which are interconnected through memoryless operators in the form

$$u_i(t) = u_{\text{ext } i}(t) - \sum_{j=1}^{m} H_{ij}(y_j(t), t) \tag{13a}$$

where f_i, h_i, H_{ij}, are all continuous. The objective of this section is to present some sufficient conditions, based on graph-theoretic techniques, for the system (13) to be well-posed. Substituting (13c) into (13b) yields

$$u_i(t) = u_{\text{ext } i}(t) - \sum_{j=1}^{m} H_{ij}(h_j(x_j(t), u_j(t), t), t),$$
$$1 = 1, \cdots, m \tag{14}$$

or, more compactly,

$$u_i(t) = u_{\text{ext } i}(t) - \sum_{j=1}^{m} E_{ij}(x_j(t), u_j(t), t),$$
$$i + i, \cdots, m. \tag{15}$$

Now, if we can "solve" the implicit equations (15) to obtain explicit expressions for u_i in the form

$$u_i(t) = C_i(x(t), u_{\text{ext}}(t), t) \tag{16}$$

where C_i is continuous, then the system (13) is well-posed.

Towards this end let us define a *dependence digraph* D, which has m vertices (the same as the number of subsystems), and contains an edge *from* vertex j *to* vertex i if and only if E_{ij} depends explicitly on u_j.

Theorem 5: If the dependence digraph D is acyclic, the system (13) is well-posed.

Proof: We shall show that if C is acyclic, then (15) can be solved in the form (17).

If D is acyclic there is at least one vertex that has no predecessor; i.e., all edges are directed away from it. Let I_0 denote the index set of vertices that have no predecessors. From the manner in which D was constructed, it follows that whenever $x \in I_0$, E_{ij} *does not* depend on u_j, for all j. Thus the ith equation in (15) is of the form

$$u_i(t) = u_{\text{ext}i}(t) - \sum_{j=1}^{m} E_{ij}(x_j(t), t), \qquad \forall i \in I_0. \tag{17}$$

Next, remove from D all edges leaving vertices $i \in I_0$, and let D_1 denote the resulting digraph. Since D is acyclic and D_1 is a subgraph of D, it is clear that D_1 is also acyclic. Let I_1 denote the index set of vertices in D_1 that do not have predecessors. Going back to D, it is clear that whenever $i \in I_1$, all predecessors of i belong to I_0. Thus, whenever $i \in I_1$, the ith equation in (15) is of the form

$$u_i(t) = u_{\text{ext}i}(t) - \sum_{j \in I_0} E_{ij}(x_j(t), u_j(t), t)$$
$$- \sum_{j \notin I_0} E_{ij}(x_j(t), t), \qquad \forall i \in I_1. \tag{18}$$

By substituting from (17) into (18), (18) can be put in the form (16). By repeating this bootstrap operation we can ultimately obtain equations of the form (16) for *all* i. Clearly all the functions C_i are continuous. ∎

This result bears some similarity to the well-posedness criterion in [4]; in particular, the graph D here is similar to the reduced digraph in [4, Th. 1]. However, this paper and [4] are concerned with two different problems.

VI. Conclusions

In this paper we have presented two types of results; namely, connections between input–output stability and Lyapunov stability, and well-posedness of interconnected systems. In the process of establishing the connections between I/O stability and Lyapunov stability, we have introduced two new types of detectability (CSA and CSB), which have the advantage that they are preserved under interconnection.

There is much work that remains to be done. First, the results of Theorems 1 and 2 can be "localized", giving a connection between "small-signal" L_p-stability and local Lyapunov stability. Second, the results presented in Section IV concerning verifications of CSA and CSB can be improved by extending them to general nonlinear systems (as opposed to linear systems with nonlinear memoryless feedback). Finally, the sufficient condition of Theorem 5 can be replaced by weaker conditions. All in all, we believe that this subject offers many challenges to future researchers.

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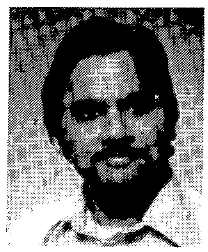
His research interests include nonlinear control and stability theory, input–output stability theory, and nonlinear programming. He is currently working on the relationship between Lyapunov stability and input–output stability.



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