Technical Notes and Correspondence

A Connective Stability Result for Interconnected Passive Systems

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Abstract—**When a large scale system is formed from linear interconnections of a number of passive subsystems, a diagonal dominance condition on the interconnection matrix is known to ensure asymptotic stability. It is shown that the same condition ensures that stability is retained when a large class of nonlinearities, including nonlinearities with memory, is allowed in the interconnections. In particular, arbitrary distributed delays may be inserted without loss of stability.**

I. Introduction

There are several known criteria—see, for example, [1], [2]—for stability of a large-scale system whose subsystems are passive. Some (but not all) of these criteria also imply connective stability in the sense of Šiljak [3]. In a crude sense, this means that stability is retained even when some or all of the interconnection gains are reduced.

A more complex but practically important problem arises when the interconnections have some memory. For example, the communication of signals from one subsystem to another might involve a time delay. Conditions under which a linear system can tolerate arbitrary time delays, without losing stability, are known [4]. For interconnected nonlinear systems, with nonlinear perturbations and time delays in the interconnections, a stability analysis is obviously more complicated.

The central result of this note is a criterion which guarantees that a large-scale system, formed by interconnecting passive subsystems, remains stable under a large class of distortions in the connections between subsystems. The class includes nonlinearities in the sector $[-1,1]$, arbitrary time delays (including distributed time delays), and many other nonlinearities with memory. The stability criterion is also necessary for absolute stability, in the sense that if it is violated, then there exist systems in the class considered which are unstable.

The result is a straightforward application of a general stability criterion in [1, Theorems 1 and 2]. If one is not particularly interested in generating Lyapunov functions then the result may also be obtained as a special case of a result by Araki [6, Theorem 4], with roughly an equal amount of effort.

II. Problem Formulation

The question treated in [4] was the stability of a set of equations of the form

$$\dot{x}_i(t) = -a_{ii}x_i(t) + \sum_{j\neq i} a_{ij}x_j(t-T_{ij}), \qquad i=1,\cdots,N \tag{1}$$

where the a_{ij} and T_{ij} are constants (actually, the case of time-varying parameters was also covered in [4], but we shall not attempt this extension). Here, a somewhat more general class of systems will be considered.

Let P_T be the causal truncation operator, and assume that all signals x_i are such that, for all $T<\infty$, P_Tx_i belongs to a Hilbert space. The notation $\langle x,y\rangle_T$ will be used to mean $\langle P_Tx,y\rangle = \langle x,P_Ty\rangle = \langle P_Tx,P_Ty\rangle$. Also $\|x\|_T$ denotes the truncated norm $\|P_Tx\|$.

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Basically, the large-scale system considered here is an interconnection of N passive systems. We have N inputs u_i and N outputs y_i, which are constrained via

$$2\langle u_i,y_i\rangle_T + \frac{1}{a_{ii}}\langle u_i,u_i\rangle_T \geq 0 \qquad \text{for all } T<\infty. \tag{2}$$

(This is actually a weaker constraint than passivity, since we will later require $a_{ii}>0$.) Apart from this constraint, the internal details of the mapping from u_i to y_i are of no interest.

The interconnection has the form

$$u_i = -a_{ii}y_i + \sum_{j\neq i} a_{ij}\psi_{ij}(y_j) \tag{3}$$

where the a_{ij} are scalar constants, and the ψ_{ij} are operators which obey the constraint

$$\|\psi_{ij}(\sigma)\|_T \leq \|\sigma\|_T \qquad \text{for all } \sigma \text{ and all } T<\infty. \tag{4}$$

This means that the ψ_{ij} can be linear gains in the range $[-1,1]$, or memoryless nonlinearities in the sector $[-1,1]$, or even elements with memory. In particular, time delays satisfy (4). This means that the class of systems described by (2)–(4) includes systems of the form (1), as a special case.

Inequalities (2) and (4) are both special cases of the inequality

$$\langle y_i,Q_iy_i\rangle_T + 2\langle y_i,S_iu_i\rangle_T + \langle u_i,R_iu_i\rangle_T \geq 0 \tag{5}$$

for all $T<\infty$ and all u_i, where u_i denotes the input to a subsystem and y_i its output. For constant matrices Q_i, S_i, R_i, we call a subsystem satisfying (5) dissipative [1], or (Q_i,S_i,R_i) dissipative when it is desirable to show the coefficients explicitly. Casting the constraints in this form allows the use of a stability test from [1], briefly described as follows.

Let M dissipative systems be interconnected with the linear constraints

$$u_i = -\sum_{j=1}^{M} h_{ij}y_j. \tag{6}$$

Let H be the matrix with elements h_{ij}. Define $Q=\text{diag}(Q_1,Q_2,\cdots,Q_M)$, $S=\text{diag}(S_1,S_2,\cdots,S_M)$, and $R=\text{diag}(R_1,R_2,\cdots,R_M)$. Then form the matrix

$$\hat{Q} = SH + H^TS^T - H^TRH - Q. \tag{7}$$

It is shown in [1] that if $\hat{Q}$ is positive definite, then the interconnected system is both input-output stable and (with some minor technical assumptions) asymptotically stable in the sense of Lyapunov. The input-output stability result is of no relevance here because the system (2)–(4) has no external inputs. For asymptotic stability, we shall assume that the smoothness and similar assumptions listed in [1] are satisfied.

Notice that the above result applies to linearly connected subsystems, whereas we have nonlinear (and not necessarily memoryless) interconnections. To get around this problem the ψ_{ij} will be treated as separate subsystems. This produces a total of N^2+N subsystems. The first N, namely, the original subsystems, are passive. The remainder are described by

$$y_i = \psi_{kl}(u_i) \qquad \text{for } i=N+1,\cdots,N^2+N$$

where the integers k and l are specified uniquely by $i=Nk+l$. (N of these subsystems, namely the ψ_{kk}, are fictitious and will not be connected to anything; but they are included to keep the notation simple). The interconnection matrix is

$$H=\left[\begin{array}{c|cccc} A^{(0)} & -A^{(1)} & -A^{(2)} & \cdots & -A^{(N)} \\ \hline -I & & & & \\ -I & & & & \\ \vdots & & & 0 & \\ -I & & & & \end{array}\right]$$

where I denotes an $N\times N$ identity matrix, $A^{(0)}=\operatorname{diag}(a_{11},a_{22},\cdots,a_{NN})$, and $A^{(k)}$ is the matrix with elements

$$A_{ij}^{(k)}=\begin{cases} a_{ij} & \text{if } i=k \text{ and } i\neq j \\ 0 & \text{otherwise.}\end{cases}$$

For $1\leqslant i\leqslant N$, subsystem i is $(0,p,a_{ii}^{-1}p)$-dissipative for any real $p>0$. Similarly the remaining subsystems are $(-p,0,p)$-dissipative for any $p>0$. This leads to

$$Q=\operatorname{diag}(0,-P^{(1)},-P^{(2)},\cdots,-P^{(N)})$$
$$S=\operatorname{diag}(P^{(0)},0,0,\cdots,0)$$
$$R=\operatorname{diag}\left(P^{(0)}(A^{(0)})^{-1},P^{(1)},P^{(2)},\cdots,P^{(N)}\right)$$

where the $P^{(k)}$ are arbitrarily chosen positive definite diagonal matrices.

III. The Stability Criterion

Before proceeding, we need the following definition.

Definition [5]: The matrix A is quasi-dominant if there exists a set of real $d_i>0$ such that

$$d_ia_{ii}>\sum_{j\neq i} d_j|a_{ij}|.$$

There is a very simple test for quasi-dominance [5]. Let the matrix $\hat{A}$ be defined by

$$\hat{a}_{ii}=a_{ii}$$
$$\hat{a}_{ij}=-|a_{ij}|, \qquad \text{for } j\neq i.$$

Then A is quasi-dominant iff all leading principal minors of $\hat{A}$ are positive.

Theorem: If A is quasi-dominant, then every system constrained by (2)–(4) is stable.

Proof: The transpose of a quasi-dominant matrix is also quasi-dominant. This means that there are real numbers $\{d_i,\ i=1,\cdots,N\}$ and $\{e_i,\ i=1,\cdots,N\}$, all positive, such that

$$d_ia_{ii}-\sum_{j\neq i} d_j|a_{ij}|=\gamma_i>0$$

and

$$e_ia_{ii}-\sum_{j\neq i} e_j|a_{ji}|=\delta_i>0.$$

Let the diagonal matrices $P^{(k)}$ be defined by

$$P_{ii}^{(0)}=\frac{e_i}{d_i}$$
$$P_{ii}^{(k)}=\frac{e_k|a_{ki}|}{d_i}+\frac{1}{N}\frac{\delta_i}{d_i} \qquad \text{for } k\neq 0,\ k\neq i$$
$$P_{ii}^{(i)}=\frac{1}{2N}\frac{\delta_i}{d_i} \qquad \text{for } i>0.$$

With this choice it is easily shown that $\hat{Q}$ in (7) is block diagonal with each block quasi-dominant and therefore positive definite. □

Quasi-dominance of A is also necessary for absolute stability, in the following sense: if A is not quasi-dominant, then there exists a system in the class described by (2)–(4) which is not asymptotically stable. A simple example is provided by letting each passive system be an integrator, and by setting $\psi_{ij}(\sigma)=\sigma\operatorname{sgn}a_{ij}$.

Although the theorem above provides for Lyapunov stability, we have not explicitly constructed a Lyapunov function. However, this can be done in hindsight using the methods of [1]. The result is a function

$$V=\sum_{i=1}^{N}\frac{e_i}{d_i}\phi_i+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq 1}}^{N}\left[\frac{e_i|a_{ij}|}{d_j}+\frac{1}{N}\frac{\delta_j}{d_j}\right]\phi_{ij}$$

where the ϕ_i are storage functions [1] for the passive systems, and the ϕ_{ij} are storage functions for the ψ_{ij} interconnection subsystems. If any interconnection is memoryless, then the corresponding ϕ_{ij} will be zero.

In the particular case where each passive system has state equation $\dot{x}_i=u_i$, and each ψ_{ij} is simply a time delay T_{ij}, we recover system (1). When the storage functions are evaluated, the result is a Lyapunov function

$$V=\sum_{i=1}^{N}\frac{e_i}{d_i}x_i^2+\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq 1}}^{N}\left[\frac{e_i|a_{ij}|}{d_j}+\frac{1}{N}\frac{\delta_j}{d_j}\right]\int_{t-T_{ij}}^{t}x_j^2(\sigma)\,d\sigma.$$

IV. Conclusions

Quasi-dominance of the interconnection matrix is a known sufficient condition for stability of interconnected passive systems [1]. It is hardly surprising that quasi-dominance is also sufficient for connective stability in the sense of [3]. It has now been shown that it is sufficient for a much stronger form of connective stability, in which the perturbations in the system may include, for example, arbitrary time delays.

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On the Lyapunov Matrix Equation

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***Abstract*—In this paper the inequality which is satisfied by the determinant of the solution of the Lyapunov matrix equation $A'Q+QA=-D$ is presented. The result makes possible a lower estimate of product eigenvalues of the matrix Q and dependence from eigenvalues of the matrices A and D. This result corresponds to those presented in [2] and [3], where an estimate of the extremal eigenvalues of the matrix Q is presented. This estimate depends on the eigenvalues of the matrices A and D.**

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