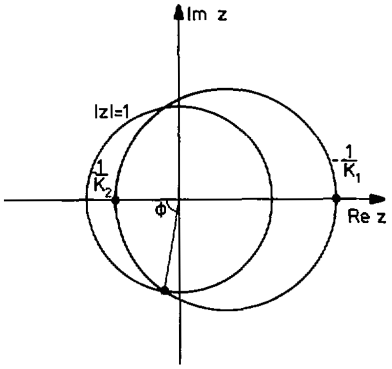

Fig. 2. Deduction of the phase-margin equation in the single-input case.

An interesting special case of Theorem 3 is $K_1=0$, in which situation one obtains the robustness sector $[0,\infty)$ by picking (A,Q) observable and solving

$$A^TP+PA=-Q.$$

$L=PB$ then yields an infinite gain margin, a 90° phase margin, and 100 percent gain reduction tolerance. These are so-called Lyapunov designs. It is possible to show that all the L's thus obtained are also linear-quadratic optimal designs, but by considering them as such, it would not be possible to guarantee this robustness sector.

4) All of the results of Theorems 1 to 3 admit an optimal control interpretation. It is easily verified using the by now standard frequency domain manipulations, which yield the results of [1] and its multivariable extensions, that using

$$J=\int_0^{\infty}(|u|^2+2u^TC^Tx+x^TMx)\,dt$$

on $\dot{x}=Ax+Bu$ with C and M such that

$$|u|^2+2u^TC^Tx+x^TMx\geqslant\gamma^2|u|^2(|\gamma|\leqslant 1)$$

will yield a robustness sector $(1/(1+|\gamma|),1/(1-|\gamma|))$. Similarly, if

$$|u|^2+2u^TC^Tx+x^TMx\leqslant\gamma^2|u|^2(|\gamma|\geqslant 1),$$

and if the optimal control problem has a solution, which is then not guaranteed, one obtains a robustness sector $(1/(1-|\gamma|),1/(1+|\gamma|))$. By multiplying the optimal gain by an appropriate factor, it is possible to obtain an arbitrary preassigned robustness sector. Theorems 1 to 3 are easily interpreted in this vein. The performance criterion is

$$\int_0^{\infty}\left(|u|^2+\frac{4K_1K_2}{(K_1+K_2)^2}(2u^TD^Tx+|D^Tx|^2+x^TQx)\right)dt$$

and the gain used is $(K_1+K_2)/2K_1K_2$ times the optimal gain.

The idea of including a cross-product term $2u^TC^Tx$ in the cost functional deserves in our opinion some attention in the linear-quadratic theory, where often only the case (u^TRu+x^TMx) with $R=R^T>0$ and $M=M^T\geqslant 0$ is treated. This extra flexibility makes it possible to generate optimal control laws that may be superior from the sensitivity or accuracy point of view. For a treatment of such optimal control problems, see [9].

5) The theory is easily generalized to the case that K_1 and K_2 become arbitrary diagonal matrices. This admits robustness design with unequal gain and phase margin requirements in the different loops.

Appendix

Proof of Proposition 1: Consider first the single-input case. The gain-margin and gain reduction tolerance assertions are immediate from the definitions. To deduce the phase-margin result, consider Fig. 2, which has been drawn for the case $K_1<0<K_2$.

It follows from the circle criterion that the condition on the Nyquist locus of $G(s)=L^T(sI-A)^{-1}B$ pertaining to condition (4) is that it be contained in a circle through $(-1/K_2,0)$ and $(-1/K_1,0)$, symmetric with respect to the real axis. This implies that the Nyquist locus intersects the unit circle at an angle at least ϕ degrees away from the negative real axis. Inserting the coordinates $(-\cos(\phi),-\sin(\phi))$ into the equation of the circle bounding the Nyquist locus yields

$$\left(-\cos(\phi)-\frac{1/K_1+1/K_2}{2}\right)^2+(-\sin(\phi))^2=\left(\frac{-1/K_1+1/K_2}{2}\right)^2$$

whence

$$\cos(\phi)=\frac{K_1K_2+1}{K_1+K_2}.$$

The multivariable case may be reduced to the above situation by using the fact that a unitary matrix is unitarily equivalent to a diagonal one. □

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Connections Between Finite-Gain and Asymptotic Stability

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***Abstract*—The relationship between input-output and Lyapunov stability properties for nonlinear systems is studied. Well-known definitions for the input-output properties of finite-gain and passivity, even with quite reasonable minimality assumptions on a state-space representation, do not necessarily imply any form of stability for the state. Attention is given to the precise versions of input-output and observability properties which guarantee asymptotic stability. Particular emphasis is given to the possibility of multiple equilibria for the dynamical system.**

I. Introduction

For causal linear time-invariant systems, there are well-known strong equivalences among a variety of definitions of stability [1]. In particular, $\mathcal{L}_2$ finite-gain stability implies, under minimality assumptions, global

Manuscript received April 12, 1979; revised September 10, 1979 and March 27, 1980. Paper recommended by M. Vidyasagar, Chairman of the Stability, Nonlinear, and Distributed Systems Committee.
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0018-9286/80/1000-0931$00.75

asymptotic stability in the sense of Lyapunov. It is tempting to suppose that a similar connection exists for a broad class of nonlinear systems.

To a certain extent this issue has been settled by a theorem owing to Willems [2]. One of the results in [2] states that a uniformly observable realization of an input-output stable dynamical system, with a reachable state-space, is globally asymptotically stable. However, Willems' definition of uniform observability is a strong one which excludes many interesting nonlinear systems—particular reference being to systems with multiple equilibria. It will be shown in this paper that, if Willems' definition of global observability is replaced by a similar local definition, then input-output stability does not necessarily imply local asymptotic stability. In fact, the system can be completely unstable in the sense of Lyapunov.

In view of this negative result, we shall restrict attention to those situations where input-output stability is taken to mean finite-gain input-output stability. One well-known definition of finite-gain [3] is

$$\|P_Ty\| \leq k\|P_Tu\| + \beta$$

in standard notation. (A precise statement is given in Definition 1.) In earlier work on input-output properties—see, for example, the treatment in [4]—the above definition was used with $\beta=0$. Generally speaking, there seems to have been no consideration given to the difference between the two definitions ($\beta=0$ or β allowed to be nonzero) except in the case of memoryless systems. A central result of this paper is that the presence of a nonzero β can make a big difference to internal stability properties of nonlinear dynamical systems.

The example (Section V) which is used in this paper to illustrate the above points arose initially from the authors' examination of a nonlinear electrical circuit. It appeared, paradoxically, that a circuit consisting of a passive nonlinear capacitor and a linear positive resistor was totally unstable. The difficulty turned out to be one of defining the term "passive" [5]. A plausible definition [3] is

$$\langle P_Tu, P_Ty\rangle \geq -\beta$$

but again there is the question of whether β must be zero, or an arbitrary real constant. It turns out that issues related to passivity and to finite-gain stability can be conveniently treated in parallel by working within the framework of so-called dissipative systems [6]–[9]. This will be the approach adopted in this paper. There are two important features of the theory of dissipative systems. First, and of vital importance to this paper, there is a connection between input-output properties and properties of a state-space representation in terms of the existence of energy-like functions of the state. These functions can often serve as Lyapunov functions. Second, the formulation of general stability results for interconnected systems is facilitated. This feature is not of concern here; the basic ideas and results have been presented elsewhere [6], [9]. However, it is worth noting that the results to be presented here also have independent interest as improved lemmas in the study of interconnected systems stability.

The structure of the paper is as follows. Section II presents notation and definitions for concepts to be studied. The setting is somewhat more general than usual insofar as it allows explicitly for the effect of different initial states on input-output behavior. Sections III and IV contain the main stability results. In Section V an example is given and Section VI looks briefly at a connection between finite-gain and exponential stability.

II. Notation and Definitions

The state-space, input signal space, and output signal space will be denoted by X, $\mathcal{U}_e$, and $\mathcal{Y}_e$, respectively (the subscript denotes that the space is an extended one in the usual sense [3], [4]). Elements of $\mathcal{U}_e$ and $\mathcal{Y}_e$ are functions of time and take their values in linear spaces U and Y, respectively. If $u\in\mathcal{U}_e$, then $u(t)\in U$ denotes the value of u at time $t\in\mathcal{T}$ (and similarly for elements of $\mathcal{Y}_e$). The time line $\mathcal{T}$ is either a semi-infinite interval $[t_0,\infty)$ on the real line, or the infinite sequence $\{t_0,t_1,\cdots\}$, depending on whether one is interested in continuous-time or discrete-time systems. The causal truncation operator P_T truncates a signal at time T; by a mild abuse of notation, P_T will be used to denote the causal truncation operator on either $\mathcal{U}_e$ or $\mathcal{Y}_e$. It is assumed that P_Tu belongs to some inner product space $\mathcal{U}$, for every $u\in\mathcal{U}_e$ and every $T\in\mathcal{T}$. A similar assumption relates elements of $\mathcal{Y}_e$ to an inner product space $\mathcal{Y}$. The notation $\langle u,v\rangle_T$ will be used as an abbreviation for $\langle P_Tu, P_Tv\rangle$, and similarly $\|u\|_T$ means $\|P_Tu\|$, where $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote inner products and norms.

The input-output mapping, assumed to be causal and invariant under time shifts, depends on the initial state $x_0\in X$. That is, we write $y=G(x_0)u$ where $G(x_0)$: $\mathcal{U}_e\to\mathcal{Y}_e$ is an operator depending on x_0. For any region $\Omega\subseteq X$, $G(\Omega)$ denotes the family $\{G(x_0): x_0\in\Omega\}$. In many applications, Ω will consist of a single point. [In such cases, we shall make no distinction between $G(\Omega)$ and $G(x_0)$.]

A complete input-output representation of a dynamical system is given by $G(X)$. We also assume a time-invariant state-space description of the dynamical system with the state transition mapping ψ: $\mathcal{T}^2\times X\times\mathcal{U}_e\to X$ and the readout mapping r: $\mathcal{T}\times X\times U\to Y$ satisfying the usual axioms [18]. These mappings describe the time evolution of the state and output according to $x(t)=\psi(t,t_0,x(t_0),u)$ and $y(t)=r(t,x(t),u(t))$. The state-space X is assumed to be a normed space.

Definition 1: $G(\Omega)$ is *weakly finite-gain stable* (WFGS) if there exists a function β: $\Omega\to R$ and a constant $k\in R$ such that

$$\|G(x_0)u\|_T \leq k\|u\|_T + \beta(x_0) \tag{1}$$

for all $u\in\mathcal{U}_e$, all $T\in\mathcal{T}$ and all $x_0\in\Omega$. If $\beta(x_0)$ is identically zero in Ω, we call $G(\Omega)$ *finite-gain stable* (FGS).

Consider the memoryless continuous linear operators Q: $\mathcal{Y}_e\to\mathcal{Y}_e$, S: $\mathcal{U}_e\to\mathcal{Y}_e$ and R: $\mathcal{U}_e\to\mathcal{U}_e$, where both Q and R are self-adjoint.

Definition 2: $G(\Omega)$ is *weakly* (Q,S,R)*-dissipative* if there exists a function β: $\Omega\to R$ such that

$$\langle G(x_0)u, QG(x_0)u\rangle_T + 2\langle G(x_0)u, Su\rangle_T + \langle u, Ru\rangle_T + \beta(x_0) \geq 0 \tag{2}$$

for all $u\in\mathcal{U}_e$, all $T\in\mathcal{T}$ and all $x_0\in\Omega$. If $\beta(x_0)$ is identically zero in Ω, we call $G(\Omega)$ (Q,S,R)*-dissipative*.

Definitions 1 and 2 generalize corresponding definitions in [3] and [9], respectively, by explicitly allowing for dependence of input-output properties on the initial state x_0. In the literature, the term "finite-gain stable" is used to mean either FGS or WFGS (without x_0 dependence). The distinction between the two will be important in this paper where the emphasis is on nonlinear systems. (For linear systems with zero initial state, there is no loss of generality in setting $\beta=0$ in (1) and (2).) The property of dissipativeness was first introduced by Willems [6]. His definition, in state-space terms, is different from the input-output definition above. However, we will see that it is essentially equivalent to the present definition of weak dissipativeness. The distinction above between "dissipative" and "weak dissipative" is new. For brevity, we will continue to use these terms with the context making clear which triple (Q,S,R) is meant.

A minor point to notice is that the spaces $\mathcal{U}_e$ and $\mathcal{Y}_e$ must be related in such a way that the inner product $\langle y, Su\rangle_T$ makes sense. If S is zero, it is possible to drop the assumption that $\mathcal{U}$ and $\mathcal{Y}$ are inner product spaces. With no inner product defined—which happens, for example, in $\mathcal{L}_p$ spaces with $p\neq 2$—one can simply replace $\langle u,u\rangle_T$ by $\|u\|_T^2$ throughout and work only with norms.

We need some notion of reachability and observability. In the definitions below, a continuous function f: $R_+\to R_+$ is called a class $\mathcal{K}$ function if $f(0)=0$, and if it is strictly monotone increasing [10]. Also, $d(x,\Omega)$ denotes the distance of x from Ω, where

$$d(x,\Omega) = \inf_{x_0\in\Omega}\|x-x_0\|.$$

Definition 3: A region $X_1\subseteq X$ is uniformly reachable from $\Omega\subseteq X$ if there exists a class $\mathcal{K}$ function δ, and for every $x_1\in X_1$ there exists $x_0\in\Omega$, finite $t_1 > t_0$, and $u\in\mathcal{U}_e$ such that $x_1=\psi(t_1,t_0,x_0,u)$ and

$$\|u\|_{t_1}^2 \leq \delta(d(x_1,\Omega)).$$

Definition 4: A region $X_1\subseteq X$ is zero-state detectable (ZSD) with respect to $\Omega\subseteq X$ if there exists a class $\mathcal{K}$ function α and a real constant $T>t_0$ such that

$$\|G(x_1)0\|_T^2 \geq \alpha(d(x_1,\Omega))$$

for all $x_1 \in X_1 - \Omega$.

Zero-state detectability may be thought of as a weak form of observability. It implies that, with zero input, the outputs resulting from initial states outside Ω are distinguishable from those resulting from initial states inside Ω. It is somewhat similar to Willems' "uniform observability" [2], except that in [2] uniform observability is a global property (with Ω effectively being a single point).

Finally, the general connection between finite-gain and internal stability properties will be in terms of asymptotic stability. Interpreting the subset Ω of X as a set of "rest states" for the system, it is natural to consider local asymptotic stability of Ω. This is stability in the sense of Lyapunov, plus the property that, with zero input, all state trajectories starting in a region containing Ω tend to Ω as time evolves [11]. When Ω contains only a single point, it becomes reasonable to consider global asymptotic stability where all trajectories starting in X tend to Ω.

III. Input-Output Results

Dissipativeness is a generalization of finite-gain stability, in the sense that $(-I,0,k^2I)$-dissipativeness is precisely equivalent to inequality (1) with $\beta=0$. The corresponding "weak" properties are not so closely linked, but one can still get the following result.

Theorem 1: If $G(\Omega)$ is weakly (Q,S,R)-dissipative with Q strictly negative definite, then $G(\Omega)$ is WFGS.

Proof: With $Q+\mu I<0$ for some scalar $\mu>0$, a straightforward use of norm inequalities similar to arguments in [9] shows that (2) implies (1), although with a different β. □

For dissipativeness, there is a stronger connection which is easily obtained.

Theorem 2: $G(\Omega)$ is (Q,S,R)-dissipative with Q strictly negative definite iff $G(\Omega)$ is FGS.

Proof: In the proof of Theorem 1, if β in (2) is identically zero, then one arrives at the version of (1) with β zero.

The converse is immediate. □

The next result is somewhat trivial, but it helps to demonstrate the significance of β.

Theorem 3: If $G(\Omega)$ is weakly (Q,S,R)-dissipative, and X_1 is uniformly reachable from Ω, then $G(X_1)$ is weakly (Q,S,R)-dissipative.

Proof: Choose any $x_1 \in X_1$, and then any $t_1>t_0$, $u\in\mathcal{U}_e$ and $x_0\in\Omega$ such that $x_1=\psi(t_1,t_0,x_0,u)$. Let $u(t)$ be arbitrary for $t>t_1$. Inequality (2) becomes, by time-invariance,

$$\langle G(x_1)u, QG(x_1)u\rangle_T + 2\langle G(x_1)u, Su\rangle_T + \langle u, Ru\rangle_T + \beta_{\text{new}}(x_1) \geq 0$$

where

$$\beta_{\text{new}}(x_1)=\beta(x_0)+\langle G(x_0)u, QG(x_0)u\rangle_{t_1} + 2\langle G(x_0)u, Su\rangle_{t_1} + \langle u, Ru\rangle_{t_1}.$$

Note that β_{new} depends only on x_1, because the u in the definition of β_{new} is the specific u chosen to transfer the state from x_0 to x_1. It follows that $G(x_1)$ is weakly dissipative. □

The significance of this theorem is most obvious when $\beta(x_0)=0$. If the system is dissipative with respect to one initial state (say, the origin), then it is *weakly* dissipative with respect to other reachable initial states. A similar statement holds, of course, in connection with finite gain. One can therefore think of β as allowing for the effect of nonzero initial states. If in addition Q is strictly negative definite, then from Theorem 1, and the definition of uniform reachability, it is not hard to show that β_{new} is bounded by a class $\mathcal{K}$ function. This fact will be used in the proof of Theorem 6.

IV. Asymptotic Stability

An important property of a (weakly) dissipative system is that it possesses a scalar-valued energy-like function, which, under certain circumstances, can act as a Lyapunov function. The first two results of this section relate to the details of this property.

Theorem 4: For some $X_1 \subseteq X$, $G(X_1)$ is weakly (Q,S,R)-dissipative iff there exists a function $\phi: X_1 \to R$, with $\phi(x) \geq 0$ for all $x\in X_1$, such that

$$\phi(x_1)+\langle y, Qy\rangle_T + 2\langle y, Su\rangle_T + \langle u, Ru\rangle_T \geq \phi(x_2) \tag{3}$$

for all $x_1\in X_1$, all $u\in\mathcal{U}_e$, and all $T\geq t_0$, where $y=G(x_1)u$ and $x_2=\psi(T,t_0,x_1,u)$.

Proof: For brevity, let

$$E(u,y,t_0,T)=\langle y,Qy\rangle_T+2\langle y,Su\rangle_T+\langle u,Ru\rangle_T. \tag{4}$$

(By time-invariance, values of E depend on the difference $(T-t_0)$ rather than on t_0 and T separately, but including both times in the notation improves the clarity of the proof.) Define

$$\phi(x_1) = -\inf_{\substack{u\in\mathcal{U}_e\\ T\geq t_1}} E(u, G(x_1)u, t_1, T).$$

Because the infimization includes the possibility $T=t_1$, it follows that $\phi(x_1)\geq 0$. For any $t_2\geq t_1$ and $T\geq t_2$, we also have

$$\phi(x_1) \geq -E(u,G(x_1)u,t_1,t_2) - E(u,G(x_2)u,t_2,T)$$

where $x_2=\psi(t_2,t_1,x_1,u)$. Because this inequality holds for all u, we have in particular

$$\phi(x_1) \geq -E(u,G(x_1)u,t_1,t_2) - \inf_{\substack{u\in\mathcal{U}_e\\ T\geq t_2}} E(u,G(x_2)u,t_2,T)$$

from which (3) follows. Inequality (2) implies that $\phi(x_1)\leq\beta(x_1)$, so

$$0 \leq \phi(x) < \infty \qquad \text{for all } x\in X_1.$$

The converse is easily seen by noting that (3) implies

$$\langle y,Qy\rangle_T+2\langle y,Su\rangle_T+\langle u,Ru\rangle_T+\phi(x_1)\geq 0. \qquad \square$$

A corresponding result for dissipativeness is as follows.

Theorem 5: Assume that $X_1\subseteq X$ is uniformly reachable from $\Omega\subseteq X$. Then $G(\Omega)$ is (Q,S,R)-dissipative iff there exists a function $\phi: X_1\to R$ satisfying the conditions of Theorem 4 plus $\phi(x)=0$ for all $x\in\Omega$.

Proof: If $G(\Omega)$ is dissipative and X_1 reachable from Ω, then Theorem 3 gives that $G(X_1)$ is weakly dissipative. Following the proof of Theorem 4, it only remains to show $\phi(x)=0$ for all $x\in\Omega$. This is implied immediately by the bounds

$$0\leq\phi(x)\leq\beta(x) \qquad \text{for all } x\in X_1.$$

Again, the converse is a direct consequence of (3). □

The function ϕ in (3) is called a storage function and, in general, is nonunique [6], [7]. It has the interpretation of stored energy since (3) provides an expression of energy balance for the system. This requires $E(u,y,t_0,T)$ to be regarded as the energy input to the system on time interval $[t_0,T]$. The original definition of dissipativeness given by Willems [6] was just that there exists a function ϕ satisfying the conditions of Theorem 4. Hence, this result establishes an equivalence between Willems' definition and the one for weak dissipativeness given here. Theorem 5 is a generalization of a result in [7].

The appealing feature of Theorems 4 and 5 is that they preserve the spirit of Kalman–Yakubovich–Popov theory for linear systems, which has found extensive application in other areas of stability theory [12]. That is, an equivalence is provided between input-output properties and state-space properties. For linear finite-dimensional passive systems, Theorem 5 with $Q=0$, $S=I$, $R=0$ collapses directly to the Kalman–Yakubovich–Popov lemma, on noting that passivity corresponds to positive-real transfer functions and using arguments similar to those in [6], [7].

Suppose now that $E(0,y,t_0,T)\leq 0$ for all y and all $T\geq t_0$. With zero input, inequality (3) shows that $\phi(x(t))$ is nonincreasing with time, and strictly decreasing given a suitable observability assumption. The result is that $\phi(x(t))$ asymptotically approaches one of its local minima, if it has any. The following theorem gives a precise connection between dissipativeness and asymptotic stability.

Theorem 6. Suppose $G(\Omega)$ is (Q,S,R)-dissipative for some strictly negative definite Q. Let $X_1=\{x\colon d(x,\Omega)<d_1\}$, for some $d_1>0$, be uniformly reachable from Ω and ZSD with respect to Ω. Then there exists some $d_2>0$ (with d_2 dependent on d_1) such that, with zero input, all state trajectories starting in $X_2=\{x: d(x,\Omega)\leq d_2\}$ remain in X_1, and asymptotically approach Ω.

Proof: For any $t\geq t_0$ and $x(t)\in X_1$, inequality (3) with $u=0$ reduces to

$$\phi(x(t+T))\leq\phi(x(t))-\mu\alpha(d(x(t),\Omega)) \tag{5}$$

where μ, as in the proof of Theorem 1, arises from negative definiteness of Q, and α and T are as introduced in the ZSD definition. Because $\phi(x(t+T))\geq 0$, this gives a lower bound on $\phi(x(t))$. Uniform reachability provides an upper bound (see the comments following Theorem 3), so that

$$\mu\alpha(d(x,\Omega))\leq\phi(x)\leq\zeta(d(x,\Omega)) \qquad \text{for all } x\in X_1$$

where α and ζ are class $\mathcal{K}$ functions. With $d_2=\zeta^{-1}(\mu\alpha(d_1))$, it follows easily that $d(x(t_0),\Omega)<d_2$ implies $d(x(t),\Omega)<d_1$ for all $t\geq t_0$.

Because $\phi(x(t))$ is nonincreasing and bounded below by zero, it converges monotonically to some limit ϕ_0. This means that for any $\varepsilon>0$ there exists $t_1\geq t_0$ such that

$$\phi_0\leq\phi(x(t))\leq\phi_0+\mu\alpha(\varepsilon) \qquad \text{for all } t\geq t_1.$$

Combining this inequality with (5), it follows that $d(x(t),\Omega)\leq\varepsilon$ for all $t\geq t_1$. □

Some comments on the proof of Theorem 6 are appropriate. One can prove that $\phi_0=0$, but this fact is irrelevant to the proof. Notice also that if Ω is not a connected region in X, then X_1 and X_2 need not be connected. This can happen when the system has multiple equilibria. An interesting feature of the proof is that standard Lyapunov theory results were not used. These require smoothness constraints on the Lyapunov functions which cannot be easily guaranteed for storage functions ϕ.

For systems which are only weakly dissipative, so that $\phi(x)$ need not be zero for $x\in\Omega$, the proof fails. This is because there is no class $\mathcal{K}$ upper bound on ϕ, so that, informally, the inequality $\phi(x(t))\leq\phi(x(t_0))$ need not describe a region containing Ω.

The important conclusions so far are that, with Q strictly negative definite and suitable reachability and observability assumptions, 1) dissipativeness implies both finite-gain stability and local asymptotic stability; and 2) weak dissipativeness implies weak finite-gain stability, but not necessarily asymptotic stability. One can also interpret these conclusions as saying that FGS implies local asymptotic stability, but WFGS does not. These results yield a considerable generalization of one for finite-dimensional nonlinear systems in [8].

It is worth pointing out that the proof of Theorem 6 does not rely on the quadratic nature of $E(\cdot,\cdot,t_0,T)$ shown in (4) except for the requirement that $E(0,y,t_0,T)\leq-\mu\|y\|_T^2$. In fact, most of the results of this paper generalize easily to situations where E is an arbitrary nonlinear functional satisfying some very weak assumptions. The exceptions are Theorems 1 and 2, which would require the relatively strong assumption

$$E(u,y,t_0,T)\leq f(\|Mu\|_T^2)-g(\|y+Nu\|_T^2)$$

for some continuous causal linear operators M and N, and functions f and g of a fairly restricted form. In any case, dissipativeness in this general sense is likely to be difficult to test, whereas there are, at least for finite-dimensional systems, computationally feasible tests for dissipativeness in the sense of our Definition 2 [7]. These observations suggest that there is little point in extending the results to the nonquadratic case.

V. Example

Consider the system with finite-dimensional state equations

$$\dot{x}=f_1(x)+f_2(x)+2\gamma u, \quad x(0)=x_0$$
$$y=f_2(x)+\gamma u$$

where γ is a scalar and the signal spaces are $\mathcal{L}^n(R_+)$. The functions $f_1(\cdot)$ and $f_2(\cdot)$ are assumed to have sufficient smoothness to ensure unique solutions, and to have the properties

1) $f_1(0)=f_2(0)=0$;
2) $f_1^T(x)f_2(x)\geq 0$ for all x;
3) $f_2(\cdot)$ is a gradient map, and there exists a constant $k\geq 0$ such that

$$\phi(x)\triangleq-\int_0^x f_2^T(\mu)\,d\mu+k\geq 0$$

for all x.

Using $\mathcal{L}_2$ norms and inner products, it turns out that

$$\gamma^2\|u\|_T^2-\|y\|_T^2=\langle f_1,f_2\rangle_T-\langle f_2,\dot{x}\rangle_T$$
$$\geq\phi(x(T))-\phi(x_0).$$

[This is, of course, inequality (3).] Writing this as

$$\|y\|_T^2\leq\gamma^2\|u\|_T^2+\phi(x_0),$$

it follows that the system is WFGS, and FGS if $\phi(x_0)=0$. Asymptotic stability can also be studied by using $\phi(x)$ as a Lyapunov function. With $u=0$, we have

$$\dot{\phi}(x)\leq-f_2^T(x)f_2(x).$$

Now consider the following three cases, with x taken to be a scalar for simplicity.

Case 1. Let $f_1(x)=-x^{q_1}$ and $f_2(x)=-\alpha x^{q_2}$, where $\alpha>0$ and q_1,q_2 are odd integers. Then

$$\phi(x)=\frac{\alpha x^{q_2+1}}{q_2+1}$$

and $\phi(0)=0$.

Case 2. Let $f_1(x)\equiv 0$ and $f_2(x)=-\alpha\sin x$ where $\alpha>0$. Then

$$\phi(x)=\alpha(1-\cos x)$$

and $\phi(x_0)=0$ for the infinite number of isolated points $x_0=\pm 2n\pi$ where $n=0,1,2,\ldots$.

Case 3. Let $f_1(x)=x$ and $f_2(x)=\alpha x/1+x^4$ where $\alpha>0$. Then

$$\phi(x)=\frac{\alpha}{2}\left(\frac{\pi}{2}-\arctan x^2\right)$$

and $\phi(x)>0$ for all finite x.

The three cases are illustrated in Fig. 1. In Cases 1 and 2, we have FGS and asymptotic stability. (For Case 2, take $\Omega=\{x_0: x_0=\pm 2n\pi\}$.) In Case 3, we have only WFGS; and the system has a unique equilibrium at $x_0=0$, which is completely unstable in the sense of Lyapunov. In all three cases, the reachability and ZSD assumptions of Theorem 6 are satisfied. This illustrates that a system which is only weakly FGS can be completely internally unstable.

VI. Exponential Stability

We now know that finite-gain stability implies local asymptotic stability. For linear finite-dimensional systems, this connection is actually an equivalence [1]. However, for nonlinear systems, it is known that even globally asymptotically stable systems need not possess input-output stability properties [13], [14]. It turns out that for a large class of systems, global exponential asymptotic stability (GEAS) does imply finite-gain.

Consider the finite-dimensional state-space representation

$$\dot{x}=f(x)+g(u) \tag{6}$$
$$y=h(x)$$

where the functions $f(\cdot)$, $g(\cdot)$, and $h(\cdot)$ are assumed to have sufficient smoothness to ensure unique solutions for $\mathcal{L}_2$ signal spaces. In particular, suppose that $f(0)=0$, $g(0)=0$, $h(0)=0$, and $f(\cdot)$ is a C^1 function with bounded gradient in the sense that $|\nabla f(x)|\leq L\ \forall x$, where L is a constant and $|\cdot|$ denotes the Euclidean norm.

Theorem 7. If the origin is GEAS for the autonomous system $\dot{x}=f(x)$ and functions $g(\cdot)$ and $h(\cdot)$ have finite-gain (with respect to Euclidean norm), then system (6) is FGS.

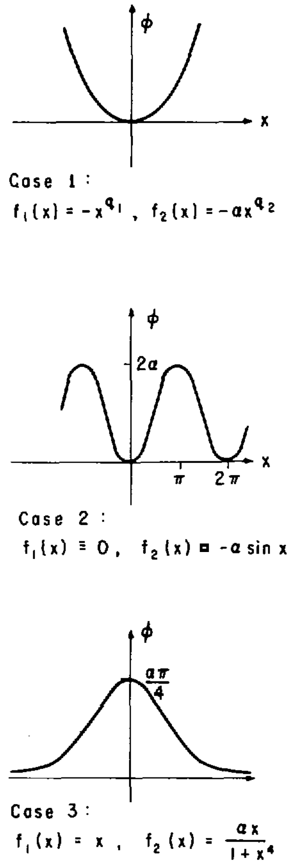

Fig. 1. Storage functions for example.

Proof: It is a known result [15] that GEAS for $\dot{x}=f(x)$, with $f(\cdot)$ possessing the above mentioned smoothness properties, implies the existence of a Lyapunov function $V: X \to R_+$ satisfying

$$\dot{V} \leqslant -c_1|x|^2 \tag{7a}$$

$$|\nabla V| \leqslant c_2|x|. \tag{7b}$$

Here $\dot{V}$ refers to the derivative along system trajectories.

For $x(0)=0$ and some $T>0$, consider

$$\int_0^T \left[\frac{c_2}{c_1}|x||g(u)|-|x|^2\right] dt \geqslant \frac{1}{c_1}\int_0^T [|\nabla V||g(u)|+\nabla V^T f(x)\, dt]$$

using (7)

$$\geqslant \frac{1}{c_1}[V(x(T))-V(x(0))]$$

$$\geqslant 0 \qquad \text{for } x(0)=0.$$

Hence

$$\int_0^T |x|^2 dt \leqslant \frac{c_2}{c_1}\int_0^T |g(u)||x|\, dt.$$

By the Schwarz inequality, this gives

$$\|x\|_T \leqslant \sqrt{\frac{c_2}{c_1}}\ \|g(u)\|_T.$$

Then finite-gain of $h(\cdot)$ and $g(\cdot)$ gives the result. □

A special case of this theorem has been useful in stability theory for interconnected systems [9]. Input-output and Lyapunov stability results feature similar stability conditions with respect to structure, but differ insofar as the isolated subsystems are typically taken to be FGS and GEAS, respectively. In [9], it is pointed out that a connection similar to that of Theorem 7 offers a considerable degree of unification to these previous results.

VII. Conclusions

The contribution of this paper is to provide some further clarification of the relationship between input-output stability and Lyapunov stability properties. All versions of input-output stability do not necessarily imply internal stability, even for reasonable minimality assumptions. Theorem 6 states conditions under which finite-gain implies local asymptotic stability. This complements a result by Willems [2], which deals only with global properties. The class of systems for which this connection is made is thus extended to include systems with multiple equilibria. Theorem 7 considers the reverse connection and gives a class of systems for which global exponential asymptotic stability implies finite-gain stability.

In the process of developing the stability results, improvements have been made to the theory of dissipative systems. Theorems 1 and 2 sharpen an important lemma in the stability theory of interconnected dissipative systems [9]. Theorems 4 and 5 represent a direct generalization of the Kalman–Yakubovich–Popov lemma. A previous version of Theorem 5 for $\mathcal{L}_2$ signal spaces provided new Lyapunov stability results for interconnected nonlinear systems [9]. This result appears to be fundamental in nonlinear systems theory. We have noted that the dissipative systems results do not require the function $E(\cdot,\cdot,t_0,T)$ to be quadratic. Very recently, Safonov and Athans [16] have considered what are essentially nonquadratic dissipation inequalities. It is interesting that they use some Lyapunov stability theory arguments in deriving input-output stability, but the development does not facilitate understanding of what internal stability properties are inherited by an input-output stable system.

Consideration of connections between input-output and Lyapunov stability properties raises some challenging problems. Solution to these are important for progress in applications. Difficulties arise in systems with multiple equilibria—for example, power systems. This area is attracting attention for applications of system theory and interest has been shown in the application of input-output methods. It has been claimed [17] that this is not possible without substantial modification to the presently known theory. The claim is based on the idea that input-output stability generally implies global asymptotic stability. The results of this paper suggest that if input-output methods are not fruitful in power systems, it will not be for that reason.

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Local Stability of Composite Systems—Frequency-Domain Condition and Estimate of the Domain of Attraction

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Abstract—**This paper is concerned with such composite systems whose subsystems contain one nonlinearity each and whose interconnections are functions of the scalar outputs of subsystems. A frequency-domain condition which assures local asymptotic stability is given under the assumptions that each nonlinearity satisfies a sector condition, that interconnections are linearly bounded, and that linear parts of subsystems may have unstable poles. In deriving the above result, such Lyapunov functions of subsystems are constructed so that their weighted sum is a Lyapunov function of the overall system. A method to estimate the domain of attraction based on the above Lyapunov functions is also studied. When the bounds on nonlinearities hold true in the entire space and when the linear parts do not have unstable poles, the present condition turns out to be the same with the L_2-stability condition which was obtained before by Araki.**

I. Introduction

The decomposition method of stability analysis is now recognized as a powerful means for the study of large-scale systems. The principal idea underlying this method was already included in the papers of Bellman [1] and Matrosov [2]. Later, especially after 1970, many researchers have derived a variety of results using this method [3]–[8]. This paper is also concerned with the application of the decomposition method to large-scale systems. Here we focus upon a specific class of systems consisting of such subsystems which have been familiar to control engineers through the absolute stability problem. Our intention exists not only in presenting a single stability condition for that class of systems but also in showing a compact model of analysis which combines the classical results obtained in the absolute stability study with the recent decomposition method. The way of analysis followed in this paper would show how to reach sharp results in other cases by applying the decomposition method, which is sometimes blamed for being "too conservative."

Thus, consider the composite system (CS) which is described by the equations

$$\dot{x}_j = F_j x_j + b_j u_j, \qquad y_j = c_j^T x_j \tag{1}$$

$$u_j = -\phi_j(y_j, t) + g_j(y_1, \cdots, y_m, t) \qquad j = 1, \cdots, m \tag{2}$$

where the scalar-valued functions $\phi_j(y_j, t)$ and $g_j(y_1, \cdots, y_m, t)$ are assumed to satisfy

$$\xi_j y_j^2 \le \phi_j(y_j, t) y_j \le \eta_j y_j^2, \qquad \phi_j(0, t) = 0 \tag{3}$$

$$|g_j(y_1, \cdots, y_m, t)| \le \sum_{k=1}^{m} \beta_{jk} |y_k| \tag{4}$$

for those values of y_j restricted by

$$-\rho_{1j} \le y_j \le \rho_{2j} \qquad j = 1, \cdots, m. \tag{5}$$

In the above, u_j and y_j are scalars; x_j, b_j, and c_j are n_j-vectors; F_j is an $n_j \times n_j$ matrix; η_j, ρ_{1j}, and ρ_{2j} are positive constants; β_{jk} is a nonnegative constant; and ξ_j is a constant less than η_j. Let x be the state vector of the CS given by

$$x = (x_1^T, \cdots, x_m^T)^T.$$

Then, from (3) and (4), the origin $x = 0$ is an equilibrium point of the CS. It is stability of this equilibrium which will be studied in the following.

If (3) and (4) are satisfied for all values of y_j without restriction (5), and if the linear parts (1) do not have unstable poles, we can use the simple frequency-domain condition of [9] to assure L_2-stability of this system. Then we can automatically assure the existence of a Lyapunov function and the global stability of the system by applying the general results about L_2-stability and existence of Lyapunov functions [22]–[25]. The purpose of this paper is to treat the case in which L_2-stability is not obtained [i.e., the case where (3) and (4) hold true only locally as assumed] and to show a concrete way of calculating the Lyapunov function and the estimate of the domain of attraction. The special case in which $\eta_j = \infty$ was already treated in [8], where the major interest lay in showing the parallelism between the Lyapunov stability analysis and the input-output stability analysis. Here, we treat the general case and also study the domain of attraction.

II. Main Results

Concerning the asymptotic stability of the CS, we have the following three theorems.

Theorem 1 (Stability Condition): Assume that the linear part (1) of each subsystem is completely controllable and observable, and let

$$f_j(s) = c_j^T (sI - F_j)^{-1} b_j. \tag{6}$$

Then, the origin $x = 0$ of the CS is asymptotically stable if there exists a positive number α_j for each j such that

$$h_j(s) = \frac{1 + (\eta_j + \alpha_j) f_j(s)}{1 + (\xi_j - \alpha_j) f_j(s)} \tag{7}$$

is positive real, and if the $m \times m$ matrix $A - B$ is an M-matrix where

$$A = \mathrm{diag}(\alpha_j), \qquad B = (\beta_{jk}) \qquad j, k = 1, \cdots, m. \tag{8}$$

Theorem 2 (Construction of Lyapunov Function): Assume the requirements of Theorem 1 are satisfied, and let δ be a sufficiently small positive number such that $A - B - \delta I$ remains an M-matrix. Then, there exist $\epsilon_j > 0$, a positive-definite $n_j \times n_j$ matrix P_j and an n_j-vector q_j which satisfy

$$F_j^T P_j + P_j F_j = -q_j q_j^T + (\xi_j - \alpha_j + \delta)(\eta_j + \alpha_j - \delta) c_j c_j^T - \epsilon_j I \tag{9}$$

$$P_j b_j - \frac{1}{2}(\xi_j + \eta_j) c_j = q_j \tag{10}$$

there exists a diagonal matrix $D = \mathrm{diag}(d_j)$ with $d_j > 0$ which makes $\tilde{M} D \tilde{M} - \tilde{\Gamma}^T D \tilde{\Gamma}$ positive semidefinite, where

$$\tilde{M} = \mathrm{diag}\left(\frac{\eta_j - \xi_j}{2} + \alpha_j - \delta\right), \qquad \tilde{\Gamma} = \mathrm{diag}\left(\frac{\eta_j - \xi_j}{2}\right) + B \qquad j = 1, \cdots, m \tag{11}$$

and the function $v(x)$ defined by

Manuscript received August 30, 1979; revised March 27, 1980. Paper recommended by M. Vidyasagar, Chairman of the Stability, Nonlinear, and Distributed Systems Committee. The work performed by M. Araki was supported by the U.S. Department of Energy under Contract EC-77-S-03-1493.

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0018-9286/80/1000-0936$00.75 ©1980 IEEE