TESTS FOR STABILITY AND INSTABILITY OF INTERCONNECTED SYSTEMS

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Abstract

This paper describes simple sufficient conditions for stability and instability of interconnections. The central theme is the notion of "dissipativeness" of the subsystems - a property which includes finite gain, passivity, conicity and some other variants as special cases.

1. Introduction

This paper is concerned with the input-output stability of interconnected systems. For brevity, we will not attempt to survey the extensive and growing literature in this field, but it is possible to make one important observation: invariably, input-output stability tests for interconnected systems start from the postulate that the behaviour of the subsystems is imperfectly known. This may well reflect physical reality; more importantly, though, it reflects a recognition that, if one used detailed equations to describe the subsystems, the resulting system description would be so complex that any stability tests so derived would be massive and unwieldy. To avoid an information explosion, the usual approach is to use only one or two parameters for each subsystem, specifying bounds on subsystem responses. The resulting stability criteria give merely sufficient conditions for stability (rather than necessary and sufficient conditions), but this is the inevitable result of trying to make the tests simple.

For example, many of the known tests are "small gain" criteria. Each subsystem is described by only one parameter, which is a bound on its input-output gain. Some other tests are based on assuming passivity, or conicity, of the subsystems. Again, these assumptions correspond to the extraction of one or two parameters to describe each subsystem.

This suggests an underlying theme to all the known input-output approaches. Our contention is that a unifying factor is the property of "dissipativeness". This property, to be defined below, includes finite gain, conicity and passivity as special cases. Consequently, one can state a general stability criterion [1] which includes many of the past published criteria as special cases.

The exposition proceeds as follows: after giving some basic definitions, it will be shown that there is a simple stability test for dissipative systems. A related instability theorem will also be given. It will then be shown that, for a composite system which is a linear interconnection of subsystems, dissipativeness of the subsystems implies dissipativeness of the overall system. This allows stability and instability criteria to be given for interconnected systems. The stability result is closely related to that in [1]; the treatment of instability is new.

There are also points of contact between the present work and that of Willems - see for example [2, 3]. Willems, who originated the dissipativeness concept, was very largely concerned with the relationship between dissipativeness and Lyapunov stability. The present approach, which follows more in the spirit of [1,4], is concerned mainly with input-output stability.

2. Definitions

Let S, S_e and T be three given spaces, where $S \subset S_e$ is a real inner product space (although in general S_e is not an inner product space). Also let $P_{(\cdot)}$ be a family of projections, such that for every $T \in T$, P_T maps S_e into S. That is, for all $u \in S_e$ and all $T \in T$, we have $P_T u \in S$.

Informally, T is the "time line", and P_T can be thought of as an operator which truncates a signal at time T. (However, we do not wish to formally restrict T to being the real line, since this would preclude application of the theory to areas like multi-dimensional filters). S_e is our signal space, and S is the space of "bounded signals". Obviously, a study of instability requires us to allow the possibility of unbounded signals - that is, signals which lie in S_e but not in S.

For any integer n, let S^n denote the space of n-tuples over S (and similarly for S_e^n). For $u = (u_1, u_2, \ldots, u_n) \in S^n$ and $v = (v_1, v_2, \ldots, v_n) \in S^n$, define

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$$\langle u, v\rangle = \sum_{i=1}^{n} \langle u_i, v_i\rangle$$

and

$$P_T u = (P_T u_1, P_T u_2, \ldots, P_T u_n)$$

Also let $\langle u, v\rangle_T$ denote $\langle P_T u, P_T v\rangle$. It is assumed that P_T has the properties $\langle P_T u, v\rangle = \langle u, P_T v\rangle = \langle P_T u, P_T v\rangle$ for all $u, v \in S_e^n$, and $||u||_T \le ||u||$ for all $u \in S^n$ and all $T \in T$ where $||u||_T$ and $||u||$ denote the square roots of $\langle u, u\rangle_T$ and $\langle u, u\rangle$ respectively.

A system G, with m inputs and p outputs, may be described as a subset of $S_e^m \times S_e^p$; that is, as a collection of possible input-output pairs (u, y) where $u \in S_e^m$ and $y \in S_e^p$.

In discussing stability, it is useful to refer to the set

$$K(G) = \{u \in S^m : y \in S^p \text{ and } (u, y) \in G\}$$

Definition: G is stable iff $K(G) = S^m$.

Definition: G is causal iff $P_T y_1 = P_T y_2$, for all $(u_1, y_1) \in G$, $(u_2, y_2) \in G$ such that $P_T u_1 = P_T u_2$, and all $T \in T$.

For the next two definitions, let $Q: S_e^p \to S_e^p$, $S : S_e^m \to S_e^p$ and $R : S_e^m \to S_e^m$ be memoryless bounded linear operators, with Q and R self-adjoint. (And with the term "memoryless" defined in terms of causality in the usual way).

Definition: G is (Q, S, R) - dissipative iff

$$\langle y, Qy\rangle_T + 2\langle y, Su\rangle_T + \langle u, Ru\rangle_T \ge 0 \qquad (1)$$

for all $T \in T$ and all $(u, y) \in G$.

Definition: G is (Q, S, R) - cyclodissipative iff

$$\langle y, Qy\rangle + 2\langle y, Su\rangle + \langle u, Ru\rangle \ge 0 \qquad (2)$$

for all $(u, y) \in G$ such that $u \in K(G)$.

With various choices of Q, S and R, it should be clear that dissipativeness embraces concepts such as passivity and conicity [5]. In particular let $Q = -k^2 I$ (where k is a scalar and I denotes the identity operator), S = 0 and R = I. Then the definition of dissipativeness reduces to $||y||_T \le k||u||_T$. In this case we say that G has finite gain, with an upper gain bound of k.

In the following two sections, we shall require that there exists $\varepsilon \ge 0$ such that $\langle y, Qy\rangle \le -\varepsilon||y||^2$ for all $y \in S^p$. If $\varepsilon = 0$, then the operator Q is nonpositive definite.

If ε is strictly positive, then Q is negative definite and has a bounded inverse. For brevity, we shall use the term "strictly negative definite" to describe an operator Q which satisfies the above inequality for some $\varepsilon > 0$.

Notice, incidentally, that if Q is negative definite then definitions (1) and (2) require the system to be unbiased (in the sense that zero input implies zero output). One way of allowing for output bias is to replace the zero on the right side of (1) and (2) by an arbitrary constant, and to define finite gain via the inequality $||y||_T \le k_1||u||_T + k_2$. To keep the notation simple, we shall ignore this possibility in the following sections.

3. Stability and Instability

The exact relationship between stability and finite gain is not entirely trivial; but to avoid digressions we shall not explore this question. It suffices here to state that finite gain is generally considered to be a strong form of stability. The "stability" results of this paper are actually "finite gain" results. Note, incidentally, that Theorem 1 is closely related to results in [1], [6].

Theorem 1: If a system G is (Q, S, R) - dissipative for some strictly negative definite Q, then it has finite gain.

Proof: If Q is strictly negative definite, then it is a relatively simple matter to manipulate the inequality (1) into the form $\langle y, y\rangle_T \le k^2\langle u, u\rangle_T$, for a scalar k depending on the norms of Q, S, R.

Theorem 2: If G is causal and (Q, S, R) - cyclodissipative for some nonpositive definite Q, but is not (Q, S, R) - dissipative, then it is not stable.

Proof: Suppose that G is causal, cyclodissipative, and stable. Then for any $u \in S_e^m$ and any $T \in T$, stability implies that $P_T u = K(G)$. Let $\bar{y}$ be an output corresponding to $P_T u$; that is, we have $(u, y) \in G$ and $(P_T u, \bar{y}) \in G$. Then (2) becomes

$$\langle \bar{y}, Q\bar{y}\rangle + 2\langle \bar{y}, SP_T u\rangle + \langle P_T u, RP_T u\rangle \ge 0$$

Now causality implies $P_T\bar{y} = P_T y$, so we have

$$\langle \bar{y}, Q\bar{y}\rangle + 2\langle y, Su\rangle_T + \langle u, Ru\rangle_T \ge 0$$

With Q nonpositive definite, this inequality implies inequality (1). That is, our original assumptions imply that G is dissipative. Conversely, if G is not dissipative, then it cannot be stable.

Notice that the above two results are not completely complementary, in that the stability result refers to finite-gain stability, but the instability result does not. Also, causality appears to be essential in Theorem 2, but is not needed in Theorem 1. The issues raised by these discrepancies have not yet been fully resolved.

4. Interconnected Systems

Suppose now we have N subsystems, such that for all i the ith subsystem G_i is dissipative or cyclodissipative with respect to some (Q_i, S_i, R_i). Let the subsystems be interconnected via

$$u_i = u_{ei} - \sum_{j=1}^{N} H_{ij} y_j, \quad i = 1, \ldots, N$$

where u_i and y_i are the input and output of G_i, the u_{ei} are external inputs, and the H_{ij} are memoryless bounded linear operators. In an obvious notation, the above constraints may be written more compactly as

$$u = u_e - Hy$$

The input and output of the overall system are taken to be u_e and y.

Let $Q = \text{diag}\{Q_1, Q_2, \ldots, Q_N\}$, $S = \text{diag}\{S_1, S_2, \ldots, S_N\}$ and $R = \text{diag}\{R_1, R_2, \ldots R_N\}$. Define the operator

$$\hat{Q} = Q + H^* R H - SH - H^* S^* \qquad (3)$$

(where the $*$ denotes adjoint).

<u>Theorem 3</u>: If all subsystems are dissipative and $\hat{Q}$ is negative definite, then the overall system has finite gain.

Proof: We have N inequalities of the form (1). By adding these, it may be shown that the overall system is $(\hat{Q}, \hat{S}, \hat{R})$ - dissipative, where $\hat{Q}$ is given by (3) and the forms of $\hat{S}$ and $\hat{R}$ are of no interest. (The details of this derivation are identical to those for the closely related result in [1]). The result then follows from Theorem 1.

<u>Theorem 4</u>: If all subsystems are cyclodissipative and unbiased in the sense that $(0, y_i) \in G_i$ implies $y_i = 0$, if $\hat{Q}$ is nonpositive definite, if the overall system is causal, and if at least one subsystem is not dissipative, then the overall system is not stable.

Proof: As in Theorem 3, it may be shown that the overall system is $(\hat{Q}, \hat{S}, \hat{R})$ - cyclodissipative, where $\hat{S} = S - H^*R$, $\hat{R} = R$, and $\hat{Q}$ is given by equation (3). Now suppose that, for some k, subsystem G_k is (Q_k, S_k, R_k) - cyclodissipative but not (Q_k, S_k, R_k) - dissipative. Then there exists a $(\bar{u}_k, \bar{y}_k) \in G_k$ and some $T \in \mathcal{T}$ such that

$$\langle \bar{y}_k, Q_k \bar{y}_k \rangle_T + 2\langle \bar{y}_k, S_k \bar{u}_k \rangle_T + \langle \bar{u}_k, R_k \bar{u}_k \rangle_T < 0$$

Now the above definitions of $\hat{Q}$, $\hat{S}$ and $\hat{R}$ are such that

$$\langle y, \hat{Q} y \rangle_T + 2\langle y, \hat{S} u_e \rangle_T + \langle u_e, \hat{R} u_e \rangle_T = \sum_{i=1}^{N} \left(\langle y_i, Q_i y_i \rangle_T + 2\langle y_i, S_i u_i \rangle_T + \langle u_i, R_i u_i \rangle_T \right)$$

(Recall that u_i and y_i are the input and output of subsystem G_i). Choose an external input u_e such that

$$u_{ei} = H_{ik} \bar{y}_k \quad \text{for } i \neq k$$

$$u_{ek} = \bar{u}_k + H_{kk} \bar{y}_k$$

With this external input there corresponds a solution u of $(I + HG)u = u_e$ such that $u_i = 0$ for $i \neq k$, and $u_k = \bar{u}_k$. The corresponding subsystem outputs are $y_i = 0$ for $i \neq k$, and $y_k = \bar{y}_k$. The above sum is therefore negative for at least one (u_e, y) pair, which suffices to show that the overall system is not $(\hat{Q}, \hat{S}, \hat{R})$ - dissipative. Instability follows from Theorem 2.

5. Discussion

In most applications, the various linear operators introduced here are matrices; so the stability tests are simply a matter of checking a matrix $\hat{Q}$ for negative definiteness. Checking the subsystems for dissipativeness or cyclodissipativeness may still be difficult, except for linear single-input single-output systems where the circle criterion [5] may be used. Criteria for dissipativeness are also known [4] for some classes of nonlinear systems.

It turns out [1] that Theorem 3 (or, more precisely, the special case of Theorem 3 which appears in [1]) includes as special cases many of the previously published stability criteria. It appears likely that Theorem 3, or something close to it, represents the limit in simplicity and generality of what one can get using the present approach. (Although, obviously, more precise stability criteria can almost certainly be obtained if one is prepared to sacrifice simplicity). The situation with respect to instability is less satisfactory, and we believe that Theorem 4 can probably be improved upon. For example, the interrelationships between causality and stability are still not fully understood. This and similar issues remain the subject of current research.

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