# Stability Criteria for Large-Scale Systems

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*Abstract*—Recent research into large-scale system stability has proceeded via two apparently unrelated approaches. For Lyapunov stability, it is assumed that the system can be broken down into a number of subsystems, and that for each subsystem one can find a Lyapunov function (or something akin to a Lyapunov function). The alternative approach is an input–output approach; stability criteria are derived by assuming that each subsystem has finite gain. The input–output method has also been applied to interconnections of passive and of conic subsystems.

This paper attempts to unify many of the previous results, by studying linear interconnections of so-called "dissipative" subsystems. A single matrix condition is given which ensures both input–output stability and Lyapunov stability. The result is then specialized to cover interconnections of some special types of dissipative systems, namely finite gain systems, passive systems, and conic systems.

## I. Introduction

ONE OF THE main thrusts in systems research over recent years has been the development of theories for large-scale systems. A significant proportion of this work has dealt with the problem of stability. The purpose of this paper is to present a new stability criterion for large-scale systems, via an approach which at the same time appears to unify many of the previously known stability results.

There is an accepted basic approach for deriving stability criteria: stability constraints are imposed on the subsystems, and then the stability properties of the composite system are deduced according to the properties of the interconnection topology. The first work along these lines is due to Bailey [1]. In this and subsequent work by others, it is assumed that a Lyapunov function, satisfying certain extra properties, exists for each subsystem. Stability of the interconnected system is studied by combining these functions in a vector and using the theory of vector Lyapunov functions—see the surveys by Matrosov [27] and Šiljak [2] on this work. An alternative approach, used by Araki and Kondo [3], and Michel and Porter [4], has been to construct a scalar Lyapunov function for the composite system as a weighted sum of the subsystem Lyapunov functions. More recently, attention has been given to input–output stability (where each subsystem is described by a mathematical relation or operator on function spaces and the methods of functional analysis are employed). Porter and Michel [5], and Cook [6] have considered the cases where the subsystems have finite gain or are conic. Their results are a natural generalization of the results for single-loop feedback systems obtained by Zames [7]. Recent extensions to the above mentioned work are numerous. To mention just some of this work, we note [8]–[10] on Lyapunov stability and [11], [12], [28], [29] where input–output stability is considered.

Although developed independently, the stability criteria arising from both methods do bear some immediate resemblance. (This is largely a superficial observation and refers to the similarity of matrix conditions restricting the interconnection structure. Some more concrete discussion along these lines is given by Cook [6].) In view of the strong parallelism between Lyapunov and input–output stability results for single-loop feedback systems, we might expect a corresponding relationship for general interconnected systems. Willems [13] has given an elegant discussion in the single-loop case within the framework of dissipative systems theory [14], [15]. Consequently, this would appear to be a good starting point for consideration of the more general situation.

In the present paper, we propose to treat input–output and Lyapunov stability side-by-side for general large-scale systems. The paper has some points of contact with Willems' work [14], [26] but whereas he was concerned with providing a general outline of how to derive stability criteria, we have concentrated on more specific aspects. The work can be considered as an outgrowth of previous work by the authors on the theory of dissipativeness and its applications to stability theory [15]–[17]. Each subsystem is assumed to have the general input–output property of dissipativeness and a very general input–output stability result is given. It is then shown that, under some further restrictions, a state space representation of the system will be asymptotically stable in the sense of Lyapunov.

Section II of the paper contains a brief treatment of dissipative systems as background material, and Section III contains the main stability results. In later sections, the results of Section III are specialized to cover some special classes of dissipative systems—namely passive, finite gain and conic systems—which have been studied extensively in the past literature. To allow a concise presentation we

Manuscript received February 8, 1977; revised October 25, 1977. This work was supported by the Australian Research Grants Committee.

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confine attention to those large-scale systems which can be viewed as a linear interconnection of (linear or nonlinear, and not necessarily finite-dimensional) subsystems, and for Lyapunov stability only time-invariant systems are considered. Apart from these limitations, the results of Sections IV–VI include as special cases all of the above cited results on input–output stability and much of the work on Lyapunov stability.

## II. Dissipative Systems

### A. An Input–Output Approach

Let $\mathcal{U}$ be an inner product space whose elements are functions $u:\mathfrak{R}\to\mathfrak{R}$. Also let $\mathcal{U}^n$ be the space of $n$-tuples (column vectors) over $\mathcal{U}$, with inner product

$$\langle u,v\rangle=\sum_{i=1}^{n}\langle u_i,v_i\rangle.$$

Then for any $u\in\mathcal{U}^n$ and any $T\in\mathfrak{R}$, a truncation $u_T$ can be defined via

$$u_T(t)=\begin{cases}u(t), & \text{for } t<T\\ 0, & \text{otherwise}.\end{cases}$$

It is also useful to speak of a "truncated inner product" $\langle u,v\rangle_T=\langle u_T,v_T\rangle$. Finally, let us define an extended space $\mathcal{U}_e^n=\{u|u_T\in\mathcal{U}^n \text{ for all } T\in\mathfrak{R}\}$.

A *system* with $m$ inputs and $p$ outputs may now be formally defined as a relation on $\mathcal{U}_e^m\times\mathcal{U}_e^p$, that is a set of pairs $(u\in\mathcal{U}_e^m, y\in\mathcal{U}_e^p)$, where $u$ is an input and $y$ the corresponding output. At this stage it is not necessary to assume that for each input there is a unique output; nor is it necessary to assume time-invariance, causality, and the like. (However such restrictions will have to be imposed when we come to consider state-space models).

Now let $Q\in\mathfrak{R}^{p\times p}$, $S\in\mathfrak{R}^{p\times m}$, and $R\in\mathfrak{R}^{m\times m}$ be constant matrices, with $Q$ and $R$ symmetric. Then we say that the above system is *$(Q,S,R)$-dissipative* if

$$\langle y,Qy\rangle_T+2\langle y,Su\rangle_T+\langle u,Ru\rangle_T\geq 0 \tag{1}$$

for all $T\in\mathfrak{R}$, and all $u$ and $y$ such that $(u,y)$ is a valid input–output pair. Related concepts—losslessness, strict dissipativeness, cyclo-dissipativeness [14]–[16]—can be defined in a similar manner; however these variants will not be needed in the present paper.

An important special case arises if we choose $Q=-I$ (where $I$ is the unit matrix of appropriate dimension), $S=0$ and $R=k^2I$, for some fixed positive real number $k$. In this case the above definition (1) reduces to

$$\|y\|_T\leq k\|u\|_T$$

where $\|\cdot\|_T$ is the truncated norm, defined via $\|x\|_T^2=\langle x,x\rangle_T$. In this case we say that the system is *finite gain input–output stable* (or, for brevity, simply *stable*) with an upper gain bound of $k$. This is a commonly used definition of input–output stability—see for example [18].

### B. A State-Space Approach

In order to study Lyapunov stability we need to introduce the concept of state. For our present purpose, an appropriate state-space model is described in the following assumption.

*Assumption 1:* There exists a metric space $X$ (the state space), a transition map $\psi:\mathfrak{R}\times\mathfrak{R}\times X\times\mathcal{U}_e^m\to X$, and a readout map $r:X\times\mathfrak{R}^m\to\mathfrak{R}^p$ such that

i) The limit $x(t)=\lim_{t_0\to-\infty}\psi(t_0,t,0,u)$ is in $X$ for all $t\in\mathfrak{R}$ and all $u\in\mathcal{U}_e^m$ (we then call $x(t)$ the state at time $t$);

ii) (Causality) $\psi(t_0,t_1,x,u_1)=\psi(t_0,t_1,x,u_2)$ for all $t_1\geq t_0$, all $x\in X$, and all $u_1,u_2\in\mathcal{U}_e^m$ such that $u_1(t)\equiv u_2(t)$ in the interval $t_0\leq t\leq t_1$;

iii) (Initial state consistency) $\psi(t_0,t_0,x_0,u)=x_0$ for all $t_0\in\mathfrak{R}$, $u\in\mathcal{U}_e^m$ and $x_0\in X$;

iv) (Semigroup property) $\psi(t_1,t_2,\psi(t_0,t_1,x_0,u),u)=\psi(t_0,t_2,x_0,u)$ for all $x_0\in X$, $u\in\mathcal{U}_e^m$ whenever $t_0\leq t_1\leq t_2$;

v) (Consistency with input–output relation) The input–output pairs $(u,y)$ are precisely those described via

$$y(t)=r\Big(\lim_{t_0\to-\infty}\psi(t_0,t,0,u),u(t)\Big);$$

vi) (Unbiasedness) $\psi(t_0,t,0,0)=0$ whenever $t\geq t_0$, and $r(0,0)=0$;

vii) (Time-invariance) $\psi(t_1+T,t_2+T,x_0,u_1)=\psi(t_1,t_2,x_0,u_2)$ for all $T\in\mathfrak{R}$, all $t_2\geq t_1$, and all $u_1,u_2\in\mathcal{U}_e^m$ such that $u_2(t)\equiv u_1(t+T)$;

viii) (Reachability) For every $x\in X$ there exists $t_0<0$ and $u\in\mathcal{U}_e^m$ such that $\psi(t_0,0,0,u)=x$.

Assumption 1 viii) is to some extent inessential, in that this assumption can be used to define $X$. Assumptions 1 vi) and vii), and even the assumption that $X$ is a metric space, can also be weakened, at the cost of complicating some of the issues raised later in this paper. The remaining parts of Assumption 1 appear to be essential for setting up a state-space model.

In defining dissipativeness for such a model, we confine attention to the choice $\mathcal{U}=\mathcal{L}_2(-\infty,\infty)$, the space of functions $u:\mathfrak{R}\to\mathfrak{R}$ which are square integrable. Inequality (1) then becomes

$$\int_{t_0}^{t_1}w(t)\,dt\geq 0$$

for all finite $t_1\geq t_0$ and all $u\in\mathcal{L}_2^m[t_0,t_1]$, whenever the initial state $x(t_0)=0$. Here, the *supply rate* $w(t)$ is given by

$$w(t)=y'(t)Qy(t)+2y'(t)Su(t)+u'(t)Ru(t). \tag{2}$$

We shall use the terms "$(Q,S,R)$-dissipative" and "dissipative with respect to supply rate (2)" interchangeably.

A crucial property of dissipative systems in state-space form is the following.

*Lemma 1:* If a system satisfying Assumption 1 is dissipative with respect to supply rate $w(t)$, then there exists a function $\phi:X\to\mathfrak{R}$ such that $\phi(0)=0,\phi(x)\geq 0$ for all $x\in X$, and

$$\phi[x(t_0)]+\int_{t_0}^{t_1} w(t)\,dt \geqslant \phi[x(t_1)]$$

for any $x(t_0)\in X$, any $u\in \mathfrak{U}_e^m$, and all $t_0,t_1\in\mathfrak{R}$ such that $t_1\geqslant t_0$. Proofs of Lemma 1 may be found in [13], [15].

In the sequel, we shall impose a further condition.

*Assumption 2:* There exists some $T>0$ and a continuous function $\alpha:\mathfrak{R}\to\mathfrak{R}$, with $\alpha(0)=0$ and $\alpha(\sigma)>0$ for all $\sigma>0$, such that with identically zero input and any initial state $x(t_0)=x_0\in X$, the output satisfies

$$\int_{t_0}^{t_0+T} y(t)'y(t)\,dt \geqslant \alpha(|x_0|)$$

where $|\cdot|$ is the metric on $X$.

Assumption 2 requires that with zero input, the output resulting from a nonzero initial state be distinguishable from that resulting from a zero initial state. That is, it is a mild observability assumption.

## III. General Stability Criteria

The stability criteria of this section refer to a linear interconnection of $N$ subsystems, each of which is dissipative (although not necessarily with respect to the same $(Q,S,R)$). The interconnection is described by

$$u_i=u_{ei}-\sum_{j=1}^{N} H_{ij}y_j, \qquad i=1,\cdots,n$$

where $u_i$ is the input to subsystem $i$, $y_i$ is its output, $u_{ei}$ is an external input, and the $H_{ij}$ are constant matrices. Writing $y=\mathrm{col}(y_1,\cdots,y_N)$, and similarly for $u$ and $u_e$, the interconnection constraint may be written more compactly as

$$u=u_e-Hy \tag{3}$$

where $H$ is a matrix whose block entries are the $H_{ij}$, with the obvious arrangement of blocks.

Let subsystem $i$ be $(Q_i,S_i,R_i)$-dissipative, and define $Q=\mathrm{diag}\{Q_1,\cdots,Q_N\}$, $S=\mathrm{diag}\{S_1,\cdots,S_N\}$ and $R=\mathrm{diag}\{R_1,\cdots,R_N\}$. Notice that $S$ need not be a square matrix, although $Q$ and $R$ are both square and indeed symmetric. Now define

$$\hat{Q}=SH+H'S'-H'RH-Q. \tag{4}$$

The matrix $\hat{Q}$ is a $p\times p$ symmetric matrix, where $p=\sum_{i=1}^{N}p_i$ and $p_i$ is the number of outputs of subsystem $i$.

*Theorem 1:* With the interconnection (3), the overall system with input $u_e$ and output $y$ is finite gain input–output stable if $\hat{Q}$ is positive definite.

*Proof:* For each subsystem, we have

$$\langle y_i,Q_iy_i\rangle_T+2\langle y_i,S_iu_i\rangle_T+\langle u_i,R_iu_i\rangle_T\geqslant 0$$

and by summing over all $i$ we obtain

$$\langle y,Qy\rangle_T+2\langle y,Su\rangle_T+\langle u,Ru\rangle_T\geqslant 0.$$

That is, the overall system, *with u considered as the input*, is $(Q,S,R)$-dissipative. From (3) and (4), this may be written as

$$\langle y,\hat{Q}y\rangle_T-2\langle y,\hat{Q}^{1/2}\hat{S}u_e\rangle_T\leqslant\langle u_e,Ru_e\rangle_T$$

where $\hat{S}=\hat{Q}^{-1/2}(S-H'R)$. Let $\alpha>0$ be a finite scalar such that $R+\hat{S}'\hat{S}\leqslant\alpha^2I$. (Obviously, such an $\alpha$ always exists). Then with a little manipulation it follows that

$$\|\hat{Q}^{1/2}y-\hat{S}u_e\|_T\leqslant\alpha\|u_e\|_T$$

and then

$$\|y\|_T\leqslant k\|u_e\|_T$$

where

$$k=\|\hat{Q}^{-1/2}\|(\alpha+\|\hat{S}\|). \qquad \nabla\nabla\nabla$$

Stability may sometimes be concluded under weaker conditions, by arguing as follows. If $\hat{Q}$ is nonnegative definite but possibly singular, a variant of the above proof can be used to show that $\langle y,\hat{Q}y\rangle_T\leqslant k_1\|u_e\|_T^2$, for some $k_1>0$. This tells us something about the boundedness of some (but not all) subsystem outputs; from (3) it might then follow (depending on the structure of $H$) that bounded $u_e$ leads to bounded $u$. Bounded $u_e$ will then lead to bounded $y$ provided that "enough" of the subsystems have bounded gain. However the precise conditions on $\hat{Q}$ and $H$ required to validate this argument are rather complicated and we shall not pursue this point further.

To obtain a state-space variant of Theorem 1, one further assumption is needed.

*Assumption 3:* The system formed from the subsystems via constraint (3) is a dynamical system satisfying Assumption 1, with state-space $\hat{X}$ equal to the Cartesian product of the state spaces of the individual subsystems.

This assumption was not needed for Theorem 1, essentially because our input–output model is general enough to encompass situations where, for example, inputs produce nonunique outputs. This and similar degeneracies are difficult to allow for in the state-space approach.

Let us now define a *Lyapunov function* as a function $V:\hat{X}\to\mathfrak{R}$, such that

i) $V(0)=0$;

ii) there exists a continuous $\alpha:\mathfrak{R}\to\mathfrak{R}$, with $\alpha(\sigma)\geqslant 0$ for all $\sigma$ and $\alpha(\sigma)=0$ iff $\sigma=0$, such that $V(x)\geqslant\alpha(|x|)$ for all $x\in\hat{X}$;

iii) with zero external input and any initial state $x(t_0)\in\hat{X}$, the subsequent state trajectory satisfies $V[x(t)]\leqslant V[x(t_0)]$ for all $t\geqslant t_0$, and $V[x(t_1)]<V[x(t_0)]$ for some finite $t_1>t_0$ unless $x(t_0)=0$.

Under mild additional conditions, the existence of a Lyapunov function implies that the origin $x=0$ of $\hat{X}$ is asymptotically stable (or even globally asymptotically stable, if condition ii) above is strengthened to require $\lim_{\sigma\to\infty}\alpha(\sigma)=\infty$). For example, asymptotic stability follows if it can be shown that for each $x(t_0)$ the subsequent trajectory is either unbounded or lies in a compact subset of $\hat{X}$ [19], as is the case for finite-dimensional systems and certain classes of infinite dimensional systems [19]. Alternatively, an assumption that $\hat{X}$ is a separable reflexive Banach space leads to weak asymptotic stability [20].

*Theorem 2:* Let each subsystem satisfy Assumptions 1 and 2, and let the interconnection (3) be such that Assumption 3 is satisfied. Then if $\hat{Q}$ is positive definite, there exists a Lyapunov function for the overall system.

*Proof:* For each subsystem, we have by Lemma 1

$$\phi_i[x_i(t_0)]+\int_{t_0}^{t_1} w_i(t)\,dt \geqslant \phi_i[x_i(t_1)], \qquad i=1,\cdots,N.$$

Now let $V(x)=\sum_{i=1}^{N}\phi_i(x_i)$. By summing the above inequalities, we obtain

$$V[x(t_0)]-\int_{t_0}^{t_1} y'\hat{Q}y\,dt \geqslant V[x(t_1)]$$

after setting $u_e \equiv 0$. From Assumption 2 and the positive definiteness of $\hat{Q}$, $V(\cdot)$ is clearly a Lyapunov function for the overall system. ∇∇∇

In view of the similarity between Theorems 1 and 2, the phrase "the system is stable" will henceforth be used to mean that the interconnected system is both finite gain input–output stable and asymptotically stable in the sense of Lyapunov. It should be recognized, however, that to deduce Lyapunov stability actually requires more assumptions than for input–output stability.

In the next three sections, we consider several important cases which lead to $\hat{Q}$ being positive definite.

## IV. Passive Systems

A passive system is one satisfying $\langle u,y\rangle_T \geqslant 0$; that is, it is $(0,I,0)$-dissipative. It is also possible to define several forms of strong passivity.

*Definition:* A $(Q,S,R)$-dissipative system is said to be

a) Passive if $Q=0, S=I, R=0$;

b) $U$-strongly passive ($U$SP) if $Q=0$, $S=I$, and $R=-\epsilon I$ for some $\epsilon>0$;

c) $Y$-strongly passive ($Y$SP) if $Q=-\epsilon I$ for some $\epsilon>0$, $S=I$ and $R=0$;

d) Very strongly passive (VSP) if $Q=-\epsilon_1 I$, $S=I$ and $R=-\epsilon_2 I$ for some $\epsilon_1>0, \epsilon_2>0$.

A $U$SP system is conventionally called strictly passive [18]. If, in addition, a $U$SP system has finite gain, then it is readily shown to be VSP.

Our first result extends a result by Sundareshan and Vidyasagar [11].

*Theorem 3:* Suppose $H+H'\geqslant 0$, and let all subsystems be passive. Let the subsystems be ordered such that the first $n_1$ are VSP, the next $n_2$ are $Y$SP, the next $n_3$ are $U$SP, and the remaining $n_4=N-(n_1+n_2+n_3)$ are passive. Let $H$ be partitioned in the obvious way as

$$H=\begin{bmatrix} H_{11} & H_{12} & H_{13} & H_{14}\\ H_{21} & H_{22} & H_{23} & H_{24}\\ H_{31} & H_{32} & H_{33} & H_{34}\\ H_{41} & H_{42} & H_{43} & H_{44}\end{bmatrix}.$$

Then a sufficient condition for stability is that the columns of

$$\begin{bmatrix} H_{13} & H_{14}\\ H_{33} & H_{34}\end{bmatrix}$$

be linearly independent.

*Proof:* We have $Q=\text{diag}\{-\Lambda_1,-\Lambda_2,0,0\}$, $S=I$, and $R=\text{diag}\{-\Lambda_3,0,-\Lambda_4,0\}$ where the $\Lambda_i$ are positive definite diagonal matrices. A short calculation then shows that $\hat{Q}$ is positive definite whenever

$$\begin{bmatrix} H_{13} & H_{14}\\ H_{33} & H_{34}\end{bmatrix}'\begin{bmatrix} \Lambda_3 & 0\\ 0 & \Lambda_4\end{bmatrix}\begin{bmatrix} H_{13} & H_{14}\\ H_{33} & H_{34}\end{bmatrix}$$

is nonsingular. ∇∇∇

After setting $n_2=n_3=0$, Theorem 3 becomes a generalization of the main result in [11].

The rank condition on $H$ above requires in effect that the outputs of the $U$SP and passive subsystems be "sufficiently well coupled" to the inputs of the VSP and $U$SP subsystems. This condition can be weakened, at the cost of replacing the condition $H+H'\geqslant 0$ by a positive definiteness condition. For simplicity, we confine attention to single-input–single-output subsystems.

*Theorem 4:* Let all subsystems be passive, but not necessarily strongly passive, and suppose that each subsystem has only one output. Then a sufficient condition for stability is that there exist a positive definite diagonal matrix $P$ such that $(PH+H'P)$ is positive definite.

*Proof:* If subsystem $i$ is $(0,1,0)$-dissipative, then clearly it is also $(0,p_i,0)$-dissipative for any $p_i>0$. Then we have $Q=0$, $S=\text{diag}\{p_1,\cdots,p_N\}$, $R=0$ and consequently $\hat{Q}=PH+H'P$. ∇∇∇

Testing for the existence of such a $P$ is discussed in the Appendix. The technique of weighting each $(Q_i,S_i,R_i)$ by a scalar $p_i>0$ can also be used in Theorem 3, but for the sake of clarity we have omitted this generalization. The technique also works for multivariable subsystems, but leads to a stability criterion which is less simple to apply than in the single-output case.

## V. A Small Gain Theorem

A large proportion of the existing stability criteria for large-scale systems, for example those in [2]–[6], [12], refer to a situation in which all subsystems have finite gain. If the interconnections are linear, then this situation fits readily into the present framework.

*Theorem 5:* Let the $i$th subsystem have finite gain $\gamma_i$, for $i=1,\cdots,N$, and suppose that each subsystem has only one input and one output. Define $\Gamma=\text{diag}\{\gamma_1,\cdots,\gamma_N\}$

and $A=\Gamma H$. Then if there exists a diagonal positive definite matrix $P$ such that

$$P-A'PA>0 \tag{5}$$

the interconnected system is stable.

*Proof:* The $i$th subsystem is $(-1,0,\gamma_i^2)$-dissipative, and therefore $(-p_i,0,p_i\gamma_i^2)$-dissipative for any $p_i>0$. Equation (4) then leads to $\hat{Q}=P-A'PA$, where $P=\operatorname{diag}\{p_1,\cdots,p_N\}$. ∇∇∇

Methods for checking the above condition (which is essentially the same problem as that which arises in checking Theorem 4) are given in the Appendix. In particular, a sufficient (but far from necessary) condition for the existence of a diagonal $P>0$ satisfying (5) is that the matrix $\hat{A}$ with elements

$$\hat{a}_{ii}=1-|a_{ii}|$$
$$\hat{a}_{ij}=-|a_{ij}|, \qquad \text{for } i\neq j$$

have positive leading principal minors. That is, it is an $M$-matrix [25]. This is precisely the criterion given in [5], [6] for input–output stability.

Similar criteria for Lyapunov stability have also appeared in many papers—see for example [2]–[4], [8]–[10]. Because the Lyapunov approach does not assume a finite gain constraint for the subsystems, but instead assumes the existence of Lyapunov functions (having certain special properties) for the subsystems, a direct comparison with those results is difficult. Actually, the assumptions made in [2]–[4] are easily shown to imply that each subsystem has finite gain; this means, at least for the case of linear interconnections and time-invariant subsystems, that the stability criteria of [3], [4] are implied by those in [5], [6] and therefore are more conservative than the present results. (The stability criterion of Šiljak [2] differs somewhat from that in [3], [4], and in general appears to give even more conservative results). However more recent versions of the Lyapunov approach, such as those in [8]–[10], rely on assumptions which are not equivalent to a finite gain constraint, and it would appear that the results of [8]–[10] neither imply nor are implied by our Theorem 5.

Incidentally, it is interesting to note that many, perhaps most, of the existing stability criteria—for example, those in [2]–[6], [10]—require an $M$-matrix test. As indicated above, use of our Theorem A2 also leads to an $M$-matrix test. The virtue of the $M$-matrix criterion is that it is easy to check, but it is normally only satisfied for *weakly coupled* subsystems. Criteria which do not require an $M$-matrix test, such as (5) above and the criteria in [8, 9] are generally more difficult to apply, but are less restrictive in terms of the degree of coupling allowed between the subsystems.

Theorem 5 may also be extended to a multi-input–multi-output situation, but unfortunately the resulting stability criterion is not as easy to check as in the single-input–single-output case.

## VI. Conic Subsystems

If a single-input–single-output system satisfies

$$\langle y-au, bu-y\rangle_T \geq 0$$

for some scalars $a$ and $b\geq a$ (and all $u$), then we say it is interior conic, or conic inside the sector $[a,b]$. Similarly if

$$\langle y-au, bu-y\rangle_T \leq 0$$

then the system is said to be exterior conic, or conic outside the sector $[a,b]$. Notice that finite gain systems are special cases of interior conic systems. Moreover, for linear systems conicity is very easily checked [7].

Suppose now that a number of conic systems are interconnected via the constraint (3). If the $i$th subsystem is inside or outside the sector $[a_i,b_i]$, then clearly it is $(-\sigma_i,\frac{1}{2}(a_i+b_i)\sigma_i,-a_ib_i\sigma_i)$-dissipative, where $\sigma_i=+1$ for internal and $-1$ for external conicity. Let $A=\operatorname{diag}\{a_1,\cdots,a_N\}$, $B=\operatorname{diag}\{b_1,\cdots,b_N\}$ and $\Sigma=\operatorname{diag}\{\sigma_1,\cdots,\sigma_N\}$. Finally, let $C=\frac{1}{2}(A+B)$ and $D=\frac{1}{2}\cdot(B-A)$.

*Theorem 6:* The above interconnection of conic subsystems is stable if there exists a positive definite diagonal $P$ such that

$$(I+CH)'P\Sigma(I+CH)-(DH)'P\Sigma(DH)>0.$$

*Proof:* After using positive weighting factors $p_i$ as in Theorems 4 and 5, we have $Q=-P\Sigma$, $S=\frac{1}{2}(A+B)P\Sigma$ and $R=-ABP\Sigma$. After a little manipulation, this leads to

$$\hat{Q}=(I+CH)'P\Sigma(I+CH)-(DH)'P\Sigma(DH). \quad \nabla\nabla\nabla$$

When the methods of the Appendix are used to check for a suitable $P$, Theorem 6 typically gives the same stability criteria as the method of Porter and Michel [5]. However, it is easy to generate examples for which Theorem 6 gives a less conservative stability criterion than the technique in [5].

## VII. Examples

In this section we consider two brief examples, in order to show how the methods of the Appendix are applied in practice.

*Example 1:* Suppose we have two finite gain subsystems, each of gain $\leq\frac{1}{2}$, and an interconnection matrix

$$H=\begin{bmatrix} 1 & -1 \\ -1 & -k \end{bmatrix}.$$

The problem is to find values of $k$ which preserve stability.

From Theorem 5, a sufficient condition for stability is that there exist a diagonal $P>0$ such that $P-A'PA>0$, where

$$A=\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & -k \end{bmatrix}=\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2}k \end{bmatrix}.$$

Applying the simplest method in the Appendix, stability follows if the matrix

$$\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\frac{1}{2}|k| \end{bmatrix}$$

has positive principal minors; that is, if $|k|<1$. This is also the result obtained by the method of [5], [6].

Alternatively, stability follows if there is a diagonal $P>0$ such that $PF+F'P>0$, where $F=(I-A)(I+A)^{-1}$. Since $F$ is a $2\times 2$ matrix, it is easily shown that this is equivalent to requiring $F$ to have positive principal minors. Applying this condition, the stability condition becomes $-1<k<\frac{5}{3}$. ∇∇∇

*Example 2:* Consider the system of Fig. 1, where the subsystems $Z_1, Z_2, Z_3$ are all passive. The interconnection matrix is

$$H=\begin{bmatrix} \alpha & 0 & -k \\ -1 & \beta & 0 \\ 0 & -1 & \gamma \end{bmatrix}.$$

Theorem 3 does not predict stability in this case, since none of the subsystems is VSP or $U$SP. However Theorem 4 predicts stability if there exists a diagonal $P>0$ such that $PH+H'P>0$. Of the methods suggested in the Appendix to check this inequality, let us consider two alternatives:

i) The inequality holds if $H$ is quasidominant. A trivial calculation shows that this is true if $\alpha>0$, $\beta>0$, $\gamma>0$ and $|k|<\alpha\beta\gamma$.

ii) Applying decision theory methods [21], [22], it is found after a tedious calculation that the necessary and sufficient condition for the existence of a suitable $P$ is $\alpha>0$, $\beta>0$, $\gamma>0$ and $-8\alpha\beta\gamma<k<\alpha\beta\gamma$.

This is the best result that can be obtained using the stability criteria of this paper. ∇∇∇

As both of these examples illustrate, the main effort in applying the results of this paper lies in deciding which of the existence criteria given in the Appendix should be used. The decision theory approach invariably gives the best results, but it also involves the greatest computational effort. Of the remaining methods, the authors have found that the quasidominance condition of Theorem A1 usually (but not invariably) gives the best results.

## VIII. Conclusions

By using the concept of dissipativeness, it has been possible to produce stability results which are valid in both a Lyapunov stability setting and an input–output setting. Moreover, the main result of this paper (Theorem 1) includes in a straightforward way many of the previously published input–output criteria. Existing Lyapunov stability criteria are not necessarily included in the present results, since they proceed from substantially different assumptions. However, it is possible to show that, in at least some of the previous works on Lyapunov stability, the assumptions imposed imply that each subsystem has finite $\mathcal{L}_2$ gain; in such cases the present paper gives simpler and more general stability criteria.

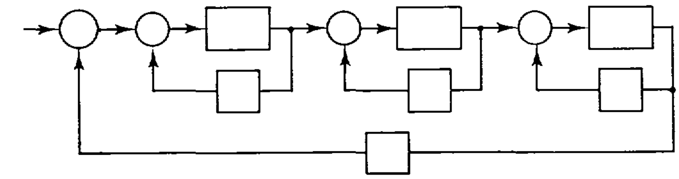

Fig. 1. Interconnection of passive systems.

The input–output stability results of this paper are formally more general than the Lyapunov stability results, in that they require fewer assumptions. On the other hand, the Lyapunov approach is frequently more useful in applications, since it provides a method of estimating stability boundaries in the case of nonglobal stability.

The techniques used in proving Theorems 1 and 2 can be extended, in a fairly obvious way, to cover nonlinearly interconnected systems. It is difficult to state a general stability criterion—in the spirit of Theorem 1—for the case of nonlinear interconnections, because of the numerous different ways in which one could place bounds on the nonlinearities. However, [30] shows one possible generalization of Theorem 2.

## Appendix

To apply the stability criteria of this paper, it is necessary to find conditions under which inequalities like

$$PF+F'P>0 \tag{A.1}$$

or

$$P-A'PA>0 \tag{A.2}$$

are satisfied for some positive definite diagonal $P$, and known $F$ or $A$. In general this is a problem of decision algebra [21], [22]; and although there are algorithms which produce necessary and sufficient existence conditions after a finite number of rational operations, these algorithms are tedious to apply. Fortunately, there are simple *sufficient* conditions for such a $P$ to exist, in terms of the quasidominance of certain matrices.

An $n\times n$ matrix $F$ is called quasidominant if there exist positive real numbers $d_1,\cdots,d_n$ such that

$$d_i f_{ii} > \sum_{j\neq i} d_j |f_{ij}| \qquad \text{for all } i.$$

Equivalently, let $\hat{F}$ be the matrix with elements

$$\hat{f}_{ii}=f_{ii}$$
$$\hat{f}_{ij}=-|f_{ij}|, \qquad \text{for } j\neq i$$

Then $F$ is quasidominant iff all leading principal minors of $\hat{F}$ are positive [23].

*Theorem A1:* If $F$ is quasidominant, then there exists a positive definite diagonal $P$ satisfying (A.1).

*Theorem A2:* If the matrix $\tilde{A}$ with elements

$$\tilde{a}_{ij} = \begin{cases} 1-|a_{ii}|, & \text{for } j=i \\ -|a_{ij}|, & \text{for } j\neq i \end{cases}$$

is quasidominant, then there exists a positive definite diagonal $P$ satisfying (A.2).

Proofs of Theorems A1 and A2 may be found in [23], or alternatively may be derived without too much difficulty from the related results in [3], [24], [25]. If the elements of $F$ or $A$ exhibit certain special sign patterns, then the quasidominance conditions become both necessary and sufficient for the existence of an appropriate $P$ [24], [25].

The inequalities (A.1) and (A.2) may also be related via the transformation $F=(I-A)(I+A)^{-1}$, $A=(I+F)^{-1}\cdot(I-F)$. Since it can happen that a matrix $F$ fails to satisfy the condition of Theorem A1 but $(I+F)^{-1}(I-F)$ satisfies the condition of Theorem A2 (or conversely), this provides a further sufficient existence condition. (Given sufficient persistence, one can extend the search even further, by trying a number of other transformations—such as $A=e^F$ or $F=(I+A)(I-A)^{-1}$—which can be used to link (A.1) and (A.2)).

More complex inequalities can be handled by similar methods. For example, the inequality

$$X'P\Sigma X - Y'P\Sigma Y > 0$$

(where $\Sigma$ and $P$ are both diagonal), can be checked via Theorem A1 after setting $F=\Sigma(X-Y)(X+Y)^{-1}$. If $\Sigma$ is a unit matrix and $X$ is invertible, then an alternative method is to set $A=YX^{-1}$ and use Theorem A2.

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