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Stability Results for Nonlinear Feedback Systems*

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A theory based on abstract energy concepts can be used to unify and extend known results on the stability of nonlinear feedback systems.

Key Word Index—Closed loop systems; Lyapunov methods; nonlinear control systems; stability criteria.

Summary—This paper presents an approach towards deriving sufficient conditions for the stability of nonlinear feedback systems. The central features of the approach are twofold. Firstly, useful stability tests are obtained for the case when the subsystems have nonlinear dynamics; secondly, a unifying set of general stability criteria are given, from which known situations can be treated as special cases and new ones are handled with equal ease. The results are obtained by use of a recently developed theory of dissipative systems.

I. INTRODUCTION

THE LITERATURE on nonlinear control system stability largely attends to the problem of finding conditions for which a linear dynamical system with nonlinear feedback is stable; historically, this work has revolved around the solution of the so-called Luré problem[1]. A major landmark in this work was the realization by Popov[2] that, for a given sector constraint on the nonlinearity, a useful restriction on the linear system, which ensures stability, can be expressed as a frequency domain condition. Subsequently, numerous other stability criteria of this nature have been developed and are now accessible in texts on stability theory—see refs [3–5, 22]. In this paper we present an approach for unifying these stability criteria, by viewing the frequency domain restrictions as special cases of a general, time domain, input–output property; furthermore, the stability criteria are formulated for situations where the dynamics belong to a broad class of nonlinear systems. The known results for linear dynamics can be easily extracted as special cases, and somewhat similar stability tests are seen to be available if the dynamics are nonlinear.

Roughly speaking, three techniques have been adopted in studying nonlinear feedback systems; these are the Lyapunov approach, the use of operator theory, and Popov's hyperstability approach. Examples of these approaches may be found in recent books on stability theory; for example, the works of Narendra and Taylor[3] and Willems[4] have adopted the Lyapunov approach, Desoer and Vidyasagar[5] consider the use of operator theory while Popov's work has been summarised in [22]. In [3], the results are obtained by using the Kalman–Yakubovich–Popov Lemma or Postive Real Lemma[6–9]. The Positive Real Lemma has strong connections with the concept of hyperstability[22]; so the two approaches have close contact. Reference [4] discusses an alternative way to derive Lyapunov functions which is based on a theory of path integrals due to Brockett[10]. The operator approach of [5] is a logical extension of the work of Zames[11]. The main theorems in [5] depend on the subsystems of the feedback system having the input–output properties of passivity, finite-gain, or conicity. Despite the generality of these results (the subsystems are considered as being causal mathematical operators and so, in general, can be infinite-dimensional, nonlinear, and time-varying), they are obviously limited in application to situations where useful criteria exist for testing such input–output properties. Only linear time-invariant systems with nonlinear feedback have been studied in this regard[4, 11]—the same class of feedback systems considered by the other methods. Indeed, the criteria obtained by all three methods are similar. Prominent amongst these are the Popov criterion and the various forms of the circle criterion.

The key to formulating a theory, using the Lyapunov stability approach, which unifies the above mentioned stability criteria was provided by Willems[12]. He proposed a definition for the property of dissipativeness for abstract dynamical systems: an abstract power input is associated with the system, and dissipativeness corresponds to the existence of a function, called a storage function, having the properties of stored energy in physical systems. This is an internal property of a system represented by state equations; however, it is easily interpreted as an input–output property for which such properties as passivity, finite gain, and conicity are special cases. In [13–15], a characterization of dissipativeness, in the input–output sense, was given for a large class of nonlinear systems, namely, those

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with time-invariant finite dimensional state equations in which the control appears linearly. This result follows as a blend of the idea of dissipativeness and work done by Anderson[9] and Moylan[13] on characterizing passivity for finite dimensional systems. For this class of dynamical systems, the storage functions are computable (in the sense that they arise as the solution of a set of algebraic equations)—an essential feature for applications. Willems[12] has suggested, in rather general terms, the usefulness of the concept of dissipativeness for studying stability. The essence of the argument is that, as energy functions, the storage functions are appealing candidates for Lyapunov functions. Some results in this direction have appeared in [14], for nonlinear systems of the above mentioned form.

It is the purpose of this paper to explore an approach, based on dissipativeness ideas, for deriving results on the stability of such nonlinear systems when subjected to feedback. The feedback can be nonlinear, and either dynamic or memoryless. General criteria are given for stability and asymptotic stability. These include, as notable special cases, Lyapunov versions of the operator results in [11] and generalizations of known Lyapunov stability criteria for the case of linear dynamics. The theorems also allow us to give conditions for stability in situations that have not been previously studied. When the dynamics are linear, frequency domain criteria can be extracted by invoking a frequency domain criterion for dissipativeness. For nonlinear dynamics, use of the stability criteria involves the solution of nonlinear algebraic equations. At least for subsystems of low dimension, this leads to stability tests which are feasible for application.

The structure of the paper is as follows. In section II, we briefly review the theory of dissipative finite dimensional systems. Sections III and IV consider general stability criteria for the cases of dynamic and memoryless feedback respectively.

II. THE THEORY OF FINITE-DIMENSIONAL DISSIPATIVE SYSTEMS

This section presents a brief review of the theory of dissipativeness for nonlinear finite-dimensional systems. A detailed treatment can be consulted in references [13–15]. The approach is related in many respects to that of Willems[12].

The systems to be studied are described by the equations

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y &= h(x) + J(x)u \end{aligned} \tag{1}$$

where x, u, and y have their values in finite-dimensional real Euclidean spaces. The functions $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ are real appropriately dimensioned functions of the state vector x, with $f(0) = h(0) = 0$. It is assumed that these functions possess some mild smoothness properties: this relates to certain existence questions in the theory and is inessential to the practical application—see ref. [14] for a more precise discussion. We also impose the restrictions that the system (1) is completely reachable (that is, for a given x_1 and t_1, there exists a $t_0 \leqq t_1$ and a locally square integrable $u(\cdot)$ such that the state can be driven from $x(t_0)=0$ to $x(t_1)=x_1$), and zero-state detectable (that is, $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $x(t) \equiv 0$). The latter assumption just says that it is possible to tell if the system is in the zero state or not, by observing the output. For linear systems, these restrictions are equivalent to complete controllability and observability.

Balakrishnan[16] has shown that, under very mild restrictions, any controllable nonlinear system with a finite-dimensional state space has a representation of the form (1).

We associate with (1) a *supply rate*

$$w(u, y) = y'Qy + 2y'Su + u'Ru \tag{2}$$

where Q, S, and R are constant matrices with Q and R symmetric. The system (1) with supply rate (2) is said to be *dissipative* if for all locally square integrable $u(\cdot)$ and all $t_1 \geqq t_0$, we have

$$\int_{t_0}^{t_1} w(t)\,\mathrm{d}t \geqq 0 \tag{3}$$

with $x(t_0)=0$ and $w(t)=w[u(t), y(t)]$ evaluated along the trajectory of (1). We impose the restriction on $w(\cdot, \cdot)$ that for any $y \neq 0$, there exists some u such that $w(u, y) < 0$; this can be seen to prevent (3) from being a trivial property.

Under these assumptions, the following result can be proved[14].

Theorem 1. The system (1) is dissipative with respect to the supply rate (2) if and only if there exist real functions $\phi(\cdot)$, $\ell(\cdot)$, and $W(\cdot)$ satisfying $\phi(x) > 0$ for all $x \neq 0$, $\phi(o) = 0$, and

$$\dot{\phi}(x) = -(\ell(x) + W(x)u)'(\ell(x) + W(x)u) + w(u, y) \tag{4}$$

along the trajectories of (1).

The function $\phi(\cdot)$ is called a *storage function* and, in general, is non-unique. Equation (4) can be viewed as a power balance equation for system (1). In [14], it is shown that (4) is equivalent to a set of algebraic equations; this facilitates the testing of a given system for dissipativeness. For linear systems dissipativeness can be checked by frequency domain techniques[11, 12, 15].

In [14] it is shown that the algebraic criterion for dissipativeness provides a technique for generating Lyapunov functions for the autonomous system $\dot{x} = f(x)$. We now turn to the extension of these ideas to feedback systems. For convenience, the special supply rates which will be used in the sequel are presented in Table 1 with the name of the corresponding form of dissipativeness.

TABLE 1. SPECIAL SUPPLY RATES

Supply rate $w(u, y)$	Type of dissipativeness
$u'y$	passive
$u'y-\varepsilon u'u$, for some $\varepsilon>0$	U – strongly passive (USP)
$u'y-\varepsilon y'y$, for some $\varepsilon>0$	Y – strongly passive (YSP)
$u'y-\varepsilon_1 u'u-\varepsilon_2 y'y$, for some $\varepsilon_1>0$ and $\varepsilon_2>0$	very strongly passive (VSP)
$u'y-y'y$	contractive
$k^2u'u-y'y$, k scalar	finite gain; k is called the upper gain bound
$y'y-\ell^2u'u$	lower bound on gain; ℓ is called the lower gain bound
$(y-au)'(bu-y), a\leqq b$	interior conic; system is said to lie inside the sector $[a, b]$
$(y-au)'(y-bu), a\leqq b$	exterior conic; system is said to lie outside the sector $[a, b]$

III. STABILITY RESULTS FOR DYNAMIC FEEDBACK

Systems of the form given in Fig. 1 will be considered throughout this paper. In this section, the subsystems H_1 and H_2 have state equations

$$\dot{x}_i=f_i(x_i)+G_i(x_i)u_i$$
$$y_i=h_i(x_i)+J_i(x_i)u_i$$

for $i=1,2$. We also assume that the feedback system is well defined; this requires the matrix $I+J_2(x_2)J_1(x_1)$ to be nonsingular for all x_1, x_2.

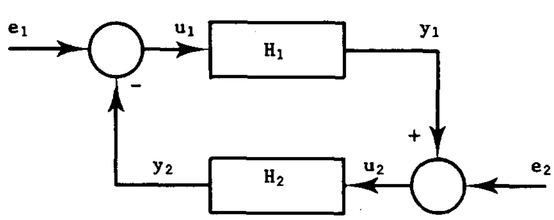

FIG. 1. Feedback configuration.

Theorem 2. Suppose that the two subsystems H_1 and H_2 are dissipative with respect to supply rates

$$w_i(u_i, y_i)=y_i'Q_iy_i+2y_i'S_iu_i+u_i'R_iu_i \qquad i=1,2$$

Then the feedback system is stable (asymptotically stable) if the matrix

$$\hat{Q}=\begin{bmatrix} Q_1+\alpha R_2 & -S_1+\alpha S_2' \\ -S_1'+\alpha S_2 & R_1+\alpha Q_2 \end{bmatrix}$$

is negative semidefinite (negative definite) for some $\alpha>0$.

Proof. Take as a Lyapunov function

$$\phi(x_1, x_2)=\phi_1(x_1)+\alpha\phi_2(x_2)$$

where ϕ_1 and ϕ_2 are the storage functions for H_1 and H_2. Then from Theorem 1, it follows that

$$\dot{\phi}\leqq w_1(u_1, y_1)+\alpha w_2(u_2, y_2)$$
$$=[y_1' \; y_2']\hat{Q}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

along the zero input trajectories of the feedback system. The second line follows by setting $u_1=-y_2$, $u_2=y_1$.

The result is now easily obtained from standard Lyapunov stability theory[17] (to conclude asymptotic stability, one can use a contradiction argument based on the LaSalle invariance principle and zero-state detectability of H_1, H_2). □

By strengthening the observability requirements, the conditions on $\hat{Q}$ can be weakened. The following theorem illustrates this point.

Theorem 3. With the same assumptions as in Theorem 2, suppose that $\hat{Q}\leqq 0$ and $S_1=\alpha S_2'$. Then the feedback system is asymptotically stable if either

(i) the matrix $(Q_1+\alpha R_2)$ is nonsingular and the composite system $H_1(-H_2)$ is zero-state detectable,

or

(ii) the matrix $(R_1+\alpha Q_2)$ is nonsingular and the composite system H_2H_1 is zero-state detectable.

Proof. In case (i), we have $\dot{\phi}\leqq 0$ with equality only if $y_1=0$. If $H_1(-H_2)$ is zero-state detectable, then $y_1(t)\equiv 0$ implies that $x_1(t)\equiv x_2(t)\equiv 0$ and we can deduce asymptotic stability by the usual contradiction argument. In case (ii), a similar argument applies. □

Theorems 2 and 3 provide stability criteria in terms of general quadratic supply rates for subsystems H_1 and H_2. We now illustrate the utility of these results by specializing to the supply rates in Table 1.

Firstly, consider H_1 and H_2 to be passive.

Corollary 1. If both H_1 and H_2 are passive, then the feedback system is stable. Asymptotic stability follows if, in addition, any one of the following (non-equivalent) conditions is satisfied:

(a) One of H_1 and H_2 is VSP.
(b) Both H_1 and H_2 are USP.
(c) Both H_1 and H_2 are YSP.
(f) $H_1(-H_2)$ is zero-state detectable, and either
  (i) H_2 is USP,
  or
  (ii) H_1 is YSP.
(e) $H_2\,H_1$ is zero-state detectable, and either
  (i) H_2 is YSP,
  or
  (ii) H_1 is USP.

Proof. Choosing $\alpha=1$, $\hat{Q}$ is of the form

$$\hat{Q}=\begin{bmatrix} -\varepsilon_{11}I-\varepsilon_{12}I & 0 \\ 0 & -\varepsilon_{21}I-\varepsilon_{22}I \end{bmatrix}$$

with the ε_{ij} either zero or positive, depending on which forms of strong passivity are assumed. By checking the various combinations, results (a)–(c) follow from Theorem 2, while (d) and (e) follow from Theorem 3.

□

This result is a Lyapunov stability version—or, more precisely, a refinement—of the positive operator theorem[11]. As given in [11], the result requires that H_1 be passive and that H_2 be USP and finite gain. It is easy to see that this corresponds to case (a) in Corollary 1.

The next result deals with the case where H_1 and H_2 have finite gain.

Corollary 2. Suppose that H_1 and H_2 have finite gain with upper gain bounds of k_1 and k_2. Then the feedback system is stable (asymptotically stable) if $k_1k_2 \leqq 1$ $(k_1k_2<1)$.

Proof. We have

$$\hat{Q}=\begin{bmatrix} (\alpha k_2^2-1)I & 0 \\ 0 & (k_1^2-\alpha)I \end{bmatrix}.$$

The result is then immediate from Theorem 2.

□

This result corresponds to the small gain theorem[11].

Another well known operator theorem[11] is based on the property of conicity; the Lyapunov version is as follows.

Corollary 3. Suppose that H_2 is inside the sector $[a+\Delta,\ b-\Delta]$, where $b>0$, and H_1 is dissipative with respect to the supply rate

$$w_1(u_1,y_1)=aby_1'y_1+(a+b)u_1'y_1 + (1-b\delta)(1+a\delta)u_1'u_1. \quad (5)$$

If both of the constants Δ and δ are zero, then the feedback system is stable. If either of these constants is zero and the other positive, with H_1 of finite gain if $a=\Delta=0$, then the feedback system is asymptotically stable.

Proof. The case where $\Delta=\delta=0$ follows immediately from Theorem 2. The details for the other cases are straightforward but somewhat tedious. To illustrate, we will now consider the case of $\delta=0$ and $\Delta>0$.

We have

$$\hat{Q}=\begin{bmatrix} (-(\alpha-1)ab-\alpha\Delta(b-a)+\alpha\Delta^2)I & \tfrac{1}{2}(\alpha-1)(a+b)I \\ \tfrac{1}{2}(\alpha-1)(a+b)I & -(\alpha-1)I \end{bmatrix}.$$

Then it can be checked that $\hat{Q}$ is negative definite if the inequalities

$$\alpha>1$$

$$\Delta(b-a-\Delta)>\frac{(\alpha-1)}{\alpha}\left(\frac{b-a}{2}\right)^2$$

are satisfied. Since $b-a\geqq 2\Delta$, this can be achieved by making $(\alpha-1)$ suitably small.

If $\Delta=0$ and $\delta>0$, the special cases of $a>0$, $a=0$, $a<0$ need to be considered separately. (Note that the finite gain specification for the case $a=\Delta=0$ is effectively just a modification of w_1.)

In each case, the result follows from Theorem 2.

■

Equation (5) requires that H_1 be interior conic if $a<0$, exterior conic if $a>0$, or that

$$H_1+\frac{1}{b}I$$

be USP if $a=0$. It is of interest to observe that the usual method of handling conic systems is to use a transformation putting the system in a form suitable for application of the small gain theorem[5, 11]; here the result followed directly from a general criterion. For the general class of systems treated in [11], Lyapunov versions of Corollaries 1 and 2 have been obtained in [18] by an alternative approach. Roughly speaking, the approach was to invoke the operator theorem to establish finite gain stability, assume that the overall system has a minimal state space representation, and use a connection between finite gain and asymptotic stability. The present approach, admittedly in a less general setting than that of [11, 18], appears to be more direct. In addition there are situations where the present approach predicts stability despite the failure of the criteria in [11] and [18].

The situations treated by Corollaries 1 to 3 have been presented because of their well-established importance; however, the following further example illustrates the ease with which the general stability criteria handle relatively unfamiliar situations.

Corollary 4. Suppose that H_1 is dissipative with respect to the supply rate $w(u_1,y_1)=u_1'y_1-m^2u_1'u_1$, m being a scalar (this implies that H_1 is USP), and H_2 has a lower gain bound of ℓ. Then the feedback system is asymptotically stable if

$$\ell>\frac{1}{m^2}.$$

Proof. We have

$$\hat{Q}=\begin{bmatrix} -\alpha\ell^2I & -\tfrac{1}{2}I \\ -\tfrac{1}{2}I & -(m^2-\alpha)I \end{bmatrix}$$

and this matrix is negative definite if

$$\ell^2 > \frac{1}{4\alpha(m^2-\alpha)}$$

and $0<\alpha<m^2$.

Choosing

$$\alpha = \frac{m^2}{2}$$

minimizes the bound on ℓ^2 and gives the required result from Theorem 2.

□

In the case where the dynamical subsystems are linear, the above results can be interpreted in terms of constraints on the transfer function matrices. This is easily achieved via a frequency domain criterion for dissipativeness[15]. Well known examples are that passive systems have positive real transfer functions and contractive systems have bounded real transfer functions[9, 19]; in [15], a criterion is given for general quadratic supply rates. For nonlinear dynamics, one has to resort to the technique in [14, 15] to test for dissipativeness.

IV. STABILITY RESULTS FOR MEMORYLESS FEEDBACK

The technique of this paper applies equally well when H_2 in Fig. 1 is memoryless. More specifically, the input–output relation for H_2 is of the form $y_2 = \psi(u_2)$ where $\psi(\cdot)$ is an unknown nonlinearity such that the feedback system is well defined. The subsystem H_1 is still assumed to have state equations of the form (1).

We first observe that Theorems 2 and 3 have direct counterparts here. To see this, suppose that H_1 is dissipative with respect to supply rate

$$w_1(u_1,y_1)=y_1'Q_1y_1+2y_1'S_1u_1+u_1'R_1u_1$$

and H_2 is dissipative in the sense that

$$w_2(u_2,y_2)=y_2'Q_2y_2+2y_2'S_2u_2+u_2'R_2u_2 \geqq 0$$

for all $u_2(\cdot)$. Taking the storage function for H_1 as a Lyapunov function, it is evident that the same conditions for stability, as were given in Theorems 2 and 3, apply here. Note, however, that in Theorem 3 the zero-state detectability requirements on the composite systems can be replaced by simple restrictions on $\psi(\cdot)$. Obviously the results in the Corollaries of Section III have counterparts here, except that we now interpret the constraints on H_2 as restricting $\psi(\cdot)$.

Our main consideration for this section of the paper is a generalized treatment of the Luré problem[1, 6, 20]. The stability of the feedback system is investigated using a Lyapunov function which is the sum of the storage function for H_1 and an integral of the nonlinear feedback. The first general criterion for stability is now presented.

Theorem 4. Suppose H_1 has no direct feedthrough (that is, $J=0$ *in* (1)) and is dissipative with respect to the supply rate

$$w_1(u_1,y_1)=y_1'Q_1y_1+2y_1'S_1u_1+u_1'R_1u_1+\dot{y}_1'Tu_1$$

for some constant matrix T. Let $\psi(\cdot)$ satisfy the conditions:

(i) $w_2(u_2,y_2)=y_2'Q_2y_2+2y_2'S_2u_2+u_2'R_2u_2\geqq 0$.

(ii) $T\psi(\cdot)$ is the gradient of a real valued function.

(iii) $\psi'T'\sigma \geqq 0$ for all σ.

Then the feedback system is stable (asymptotically stable) if $\hat{Q}\leqq 0$ ($\hat{Q}<0$), where $\hat{Q}$ is the matrix introduced in Theorem 2.

Proof. The given dissipativeness of H_1 implies that the system

$$\dot{x}=f(x)+G(x)u$$
$$y=\begin{bmatrix} h(x) \\ \dfrac{\partial h(x)}{\partial x}f(x)\end{bmatrix}+\begin{bmatrix} 0 \\ \dfrac{\partial h(x)}{\partial x}G(x)\end{bmatrix}u \qquad (6)$$

is dissipative with respect to

$$w_1(u,y)=y'\begin{bmatrix} Q_1 & 0 \\ 0 & 0\end{bmatrix}y+2y'\begin{bmatrix} S_1 \\ \frac{1}{2}T\end{bmatrix}u+u'R_1u.$$

Let $\phi(\cdot)$ be the corresponding storage function; now let us take

$$V(x)=\phi(x)+\textstyle\int_0^{h(x)}\psi'(\sigma)T'\mathrm{d}\sigma$$

as a Lyapunov function. Assumptions (ii), (iii) ensure that the integral is positive and well defined[21]. Then from Theorem 1 and assumption (i), we have

$$\dot{V}\leqq w_1(u_1,y_1)+\alpha w_2(u_2,y_2)+\psi'(y_1)T'\dot{y}_1 \quad \text{where} \quad \alpha>0$$

$$=[y_1' \; y_2']\hat{Q}\begin{bmatrix} y_1 \\ y_2\end{bmatrix} \quad \text{using} \quad u_1=-\psi(y_1).$$

The result then follows from the same arguments as were used in the proof of Theorem 2.

□

Note that the conditions on $\hat{Q}$ in Theorem 4 are the same as those for Theorem 2. Similarly, the following result corresponds to Theorem 3 and is proved using the same sort of arguments.

Theorem 5. With the same assumptions as in Theorem 4, suppose that $\hat{Q}\leqq 0$ and $S_1=\alpha S_2'$. Then the feedback system is asymptotically stable if either

(i) the matrix $(Q_1+\alpha R_2)$ is nonsingular and $\psi(0)=0$,

or

(ii) the matrix $(R_1+\alpha Q_2)$ is nonsingular and $\psi(\sigma)=0$ implies that $\sigma=0$.

D

For a particular feedback system, Theorems 4 and 5 will give a stronger stability assessment than the direct counterparts of Theorems 2 and 3 if a T can be found which makes the supply rate w_1 less restrictive on H_1 than one of the usual form.

The following result is a generalization of the Popov theorem for multivariable linear systems with nonlinear feedback[8, 20].

Corollary 5. Suppose H_1 has $J=0$, $y_1+q\dot{y}_1=0$ is not satisfied for zero input, and this system is dissipative with respect to the supply rate

$$w_1(u_1, y_1) = y_1'u_1 + u_1'Ku_1 + q\dot{y}_1'u_1, \qquad q>0.$$

Let $\psi(\cdot)$ be the gradient of a real valued function and satisfy

$$\psi(0)=0$$

$$\psi'(\sigma)\sigma - \psi'(\sigma)K\psi(\sigma) \geqq \varepsilon\sigma'\sigma \quad \text{for all} \quad \sigma \neq 0$$

where K is a nonnegative definite matrix. Then the feedback system is asymptotically stable (stable if $\varepsilon = 0$).

Proof. We have

$$\hat{Q} = \begin{bmatrix} -\varepsilon I & 0 \\ 0 & 0 \end{bmatrix}$$

and the result follows from Theorems 4, 5. □

As in Section III, there are obviously an indefinite number of other special cases we could consider.

The above results account only for the case of $\psi(\cdot)$ being independent of time. For the time-varying case, the conditions that have been derived here give uniform stability; further restrictions on $h(x)$ are needed to achieve (uniform) asymptotic stability[4].

The results of this section generalize presently known frequency domain criteria which refer to the case where H_1 is linear. Indeed, use of the frequency domain condition for dissipativeness in [15], in conjunction with the memoryless feedback versions of Corollary 3 and Corollary 5, gives easily the circle and Popov criteria respectively—these are generally accepted as the main frequency domain criteria[3–5].

V. CONCLUSIONS

The principal contribution of this paper has been to present general stability criteria for the stability of feedback systems in which both subsystems are nonlinear and time-invariant. These results contain, as special cases, presently known criteria which deal with the case of linear dynamics.

It would be of interest to explore these ideas for discrete time, time-varying stochastic feedback systems. Another important extension is in the direction of more general classes of interconnected systems. Also the theory of [15] gives the possibility of presenting corresponding instability results. Work on these topics is currently in preparation by the authors and will be reported separately.

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