Stable Inversion of Linear Systems

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Abstract—**A new and computationally efficient algorithm for inversion of linear time-invariant systems is presented. Existence conditions for either left- or right-inverse systems are also presented together with stability criteria. These criteria indicate that the algorithm will find a stable inverse whenever one exists. The results apply to both left and right inversion of a linear system and include the special case of linear finite automata or convolutional encoders.**

I. Introduction

The concept of system inversion has been around for some considerable time, with obvious applications in areas such as control theory, filtering, and coding theory. For linear systems with only a single input and output, the inversion problem is trivial, and is most easily handled using transfer function concepts. For multivariable systems the problem is more complex, both theoretically and computationally.

The earliest systematic approach to the inversion problem appears to be that of Brockett and Mesarovic [1]. In [1] the concept of *functional reproducibility*, which in broad terms means the ability to follow a given reference signal, was introduced. Reference [1] gave a criterion for functional reproducibility and subsequently Brockett [2] showed how to construct the inverse for scalar systems. Simplified criteria, and more importantly methods for constructing the inverse in the multivariable case, were introduced by Massey and Sain [3], [4], Dorato [5], and Orner [6]. An interesting development in [4] was the introduction of the idea of "inherent integration" of a linear system, which led to bounds on the number of differentiations required in the inverse system (see, also, Willsky [7]).

An important development, appearing at about the same time as the Massey and Sain papers [3], [4], was the introduction of Silverman's "structure algorithm" [8], [9]. Apart from his treatment of time-varying systems, Silverman's approach was important in two respects. First, there was no explicit test for invertibility, but rather it was shown that a system was invertible if and only if the inversion algorithm terminated in a certain way. This made the overall procedure computationally efficient. Second, it was shown in [8] and [9] how the order of the inverse system could be reduced at the end of the algorithm. The final inverse is probably of the least possible order. A drawback of the algorithm was that it could only handle systems with equal numbers of inputs and outputs, but this has since been rectified by Silverman and Payne [10]. Comparisons of the Silverman and Sain–Massey approaches may be found in [11] and [12].

Two related inversion algorithms have been given by Porter [13], [14]. The algorithm of [14] is interesting since it contained the novel idea of reducing the system order at each step of the algorithm. This was done by relating the problem of inverting a given system to that of inverting a related system with fewer states and possibly also with fewer inputs and outputs.

A point which has not yet been mentioned is that there are really two important classes of inversion problems. Briefly, a left inverse for a given system S is a system S^L which computes the input to S from knowledge of its output. A right inverse, on the other hand, is a system S^R which computes the input required in order that S have a certain desired output. If both S^L and S^R exist, then they are identical and the above distinction need not be made. (This happens only when S has the same number of inputs as outputs.) Many of the references [1]–[14] treated only the case where the input and output spaces have the same dimension, so that the distinction between left and right inverses was not highlighted. However, [1], [2], and [14] were clearly looking for a right inverse, while [3], [4], and [6] were concerned with left inverses. Silverman and Payne [10] and to some extent also Sain and Massey [4] covered both cases.

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The question of stability of an inverse has caused some difficulty. Massey and Sain [3] were probably the first to point out the problem, since they showed for one example that their algorithm produced an unstable left inverse where a stable left inverse was known to exist. A stability criterion has been given by Moore and Silverman [16], but unfortunately only for a restricted class of systems. The best results to date are those of Bengtsson [17], who proved inter alia that if any stable inverse exists, then the minimal (left or right) inverse is stable. Comments by Forney [18] and Moore and Silverman [29] are also relevant.

The main aim of this paper is to describe a computationally efficient inversion algorithm. Of all the methods discussed, it is probably closest to Silverman's algorithm [9], although it shares with Porter's method [14] the feature of dimensionality reduction at each step. It is applicable to both left and right inversion, and is believed to be computationally the most efficient method yet available. At the same time, new invertibility criteria, including stability criteria, are presented. These are superficially similar to the criteria of Dorato [5] and Wang and Davison [22], but differ in one important respect: the criteria of [5] and [22] are criteria for invertibility of a transfer function matrix, whereas those of the present paper relate to invertibility of a system specified by a set of state equations. (It will later be shown that these two sets of conditions are equivalent, but it is not *a priori* obvious that the state-space and transfer function approaches should lead to the same answer; in fact a theorem of Brockett and Mesarovic [1] to this effect is in error.)

In the initial part of the paper only left inverses are considered, and for brevity the word "inverse" will be taken to mean "left inverse" except where a right inverse is explicitly referred to.

II. Invertibility Criteria

The systems to be considered in this paper are linear and time-invariant, with state equations

$$\begin{aligned} \dot{x} &= Ax + Bu, \qquad x(0) \text{ given} \\ y &= Cx + Du \end{aligned} \tag{1}$$

where x is an n-vector, u is an m-vector, and y is a p-vector. All vectors and matrices have elements in some field F, which for the bulk of the paper will be taken to be the real field. For technical reasons it will be assumed that elements of u and y are at least n times differentiable, or alternatively that the input and output spaces are closed under differentiation (for example, if step functions are allowed as admissible inputs, then one must also allow delta functions.) Apart from this consideration the choice of signal spaces is of no great consequence in what follows.

Definition: Let u_1 and u_2 be any two inputs to the system (1), and let y_1 and y_2 be the corresponding outputs [for the same $x(0)$]. The system will be said to be (left) invertible if $y_1(t)=y_2(t)$ for all $t \geqslant 0$ implies that $u_1(t)=u_2(t)$ for all $t \geqslant 0$.

If the system is invertible, then it turns out to be possible to construct an "inverse system" whose input is the output of (1) and whose output is u. As might be expected from the corresponding frequency-domain inversion problem, the (nonunique) inverse system is a linear system possibly containing some differentiators. (The use of differentiators is unfortunately unavoidable, except in some special cases.)

It is desirable, of course, that the inverse system be asymptotically stable. The existence of stable inverses is the subject of Theorem 2 of this paper.

In the sequel it will be assumed that (1) is completely controllable and completely observable [23]. This assumption is invoked more for clarity of exposition than for any other reason, since the changes required for the more general case are relatively minor.

In the remainder of this paper, we will be concerned with the rank properties of the matrix

$$M(\lambda) \triangleq \begin{bmatrix} A-\lambda I & B \\ C & D \end{bmatrix}.$$

Apart from a sign change that is of no great significance, this matrix may be recognized as Rosenbrock's [21] system matrix. It has also been used by Wolovich [24] and Davison and Wang [30], [31] to define "transmission zeros" for a linear multivariable system. Using the results in [21], [24], and [29]–[31], it is possible to establish numerous connections between the present results and transfer-function approaches to inversion.

Our first result is as follows.

Theorem 1: The system (1) is invertible if and only if rank $M(\lambda)=n+m$ for some real λ.

Proof of Necessity: Supposing that rank $M(\lambda)<n+m$ for all real λ, it is possible to choose a set $\{\lambda_i, i=1,\cdots,l\}$—where l can be as large as desired—of distinct scalars such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_i \\ u_i \end{bmatrix}=\begin{bmatrix} \lambda_i x_i \\ 0 \end{bmatrix}$$

for vectors x_i, u_i, at least one of which is nonzero. From this observation it is possible to construct a control of the form

$$u(t)=\sum_{i=1}^{l}\alpha_i e^{\lambda_i t}u_i$$

(for some set of scalars α_i) which leads to an identically zero output when $x(0)=0$. A more complete proof may be found in [28]. ∇∇∇

The sufficiency of the condition rank $M(\lambda)=n+m$, for some real λ, for the existence of an inverse will be shown in Section III where an inverse will actually be constructed. Before proceeding, however, it is of interest to note the following stability result.

Theorem 2: System (1) possesses an asymptotically stable left inverse if and only if rank $M(\lambda)=n+m$ for all complex λ in the half-plane $\mathrm{Re}(\lambda)\geq 0$.

Proof of Necessity: Let λ_0 be such that the columns of $M(\lambda_0)$ are not linearly independent. Then there exist (possibly complex) vectors x_0 and u_0 such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}=\begin{bmatrix} \lambda_0 x_0 \\ 0 \end{bmatrix}.$$

One can then argue, as in Theorem 1, that there exists a control of the form

$$u(t)=\mathrm{Re}(e^{\lambda_0 t}u_0)$$

which produces zero output. (However, some care is needed here in treating initial conditions. For more complete details, see [28].) It follows that the inverse system—if it exists at all—must have a zero-input response of the above form, for some initial state. If $\mathrm{Re}(\lambda_0)\geq 0$, this means that there cannot exist any stable inverse system. ∇∇∇

Again, the proof of sufficiency will be deferred until Section III.

Recall that $x(0)$ was assumed to be available when constructing the inverse system. In the following section it will be found that, in general, some but not all components of $x(0)$ will be needed. In particular, we say that (1) is *invertible with unknown initial state* if the inverse can be constructed with no knowledge of $x(0)$. (This concept has been explored in some detail by Bengtsson [17].)

Theorem 3: System (1) is invertible with unknown initial state if and only if rank $M(\lambda)=n+m$ for all complex λ.

Proof of Necessity: As in Theorem 2, let λ_0, $u_0\neq 0$, and x_0 be such that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x_0 \\ u_0 \end{bmatrix}=\begin{bmatrix} \lambda_0 x_0 \\ 0 \end{bmatrix}.$$

Then an almost identical argument to that of Theorem 2 shows that the system response for $x(0)=\mathrm{Re}(x_0)$ and

$$u(t)=\mathrm{Re}(e^{\lambda_0 t}u_0)$$

is indistinguishable from that for $x(0)=0$ and $u(t)\equiv 0$, $t\geq 0$. ∇∇∇

III. Construction of the Inverse

The sufficiency proof of Theorems 1–3 proceeds by construction of the inverse system. In this section an inversion algorithm will be described, followed by a proof that the algorithm terminates in the desired manner whenever the hypotheses of Theorems 1–3 are satisfied.

It will prove convenient in the subsequent development to consider the inversion of the slightly more general system

$$\begin{aligned} \dot{x}&=Ax+Bu+v \\ y&=Cx+Du \end{aligned} \tag{2}$$

where v is a known input. The introduction of v is primarily for consistency of notation, and in most cases $v(t)$ will be zero for all t.

The algorithm proceeds by a sequence of reductions. If the system (2) initially has n states and p outputs, then in one step of the algorithm the measurements y (and v) are processed to produce a new system with $\hat{n}\leq n$ states and $\hat{p}\leq p$ outputs. At least one of these inequalities is strict, so that the inversion problem reduces to that for a smaller system. This process is repeated until the new D matrix has rank equal to m, the number of inputs. At that point, of course, the inversion problem becomes straightforward.

Assume initially that rank $D<p$. (The implications of the case rank $D=p$ will be discussed later.) Then the algorithm proceeds as follows.

A. Output Basis Change

Since D has rank $r<p$, there exists a nonsingular $p\times p$ matrix S_1 such that

$$S_1 D=\begin{bmatrix} D_0 \\ 0 \end{bmatrix}\begin{matrix} \updownarrow\ r \\ \updownarrow\ p-r \end{matrix}$$

where the $r\times m$ matrix D_0 has linearly independent rows. Partition S_1C as

$$S_1 C=\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}\begin{matrix} \updownarrow\ r \\ \updownarrow\ p-r \end{matrix}$$

where C_1 is $r\times n$. The matrix C_2 has dimensions $(p-r)\times n$; let its rank be denoted by q.

Now let a nonsingular $(p-r)\times(p-r)$ matrix S_2 be defined such that

$$S_2 C_2=\underbrace{\begin{bmatrix} \tilde{C}_2 \\ 0 \end{bmatrix}}_{\leftarrow n \rightarrow}\begin{matrix} \updownarrow\ q \\ \updownarrow\ p-r-q \end{matrix}$$

where the $q\times n$ matrix $\tilde{C}_2$ has linearly independent rows. Finally define the product

$$S=\begin{bmatrix} I & 0 \\ 0 & S_2 \end{bmatrix}S_1$$

and the transformation

$$\tilde{y}=Sy.$$

Let $\tilde{y}$ be partitioned in the obvious way as $[y_1' y_2' y_3']'$, then the output equation has the form

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}=\begin{bmatrix} C_1 \\ C_2 \\ 0 \end{bmatrix}x+\begin{bmatrix} D_0 \\ 0 \\ 0 \end{bmatrix}u.$$

The components of y_3 clearly carry no information, and may be discarded. The overall effect of this step is depicted in Fig. 1.

B. State-Space Basis Change

Because the rows of $\tilde{C}_2$ are linearly independent, $\tilde{C}_2$ has rank q and so there exists an $n\times n$ nonsingular matrix T such that

$$\tilde{C}_2 T^{-1}=\begin{array}{c} \begin{matrix} \overset{n-q}{\leftrightarrow} & \overset{q}{\leftrightarrow} \end{matrix} \\ \begin{bmatrix} 0 & I \end{bmatrix} \end{array}\updownarrow q$$

(where the unit matrix has dimension $q\times q$). With the new choice of

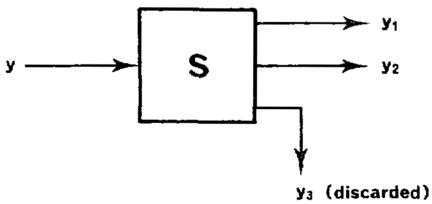

Fig. 1.

state vector $\tilde{x}=Tx$, the state equations become

$$\dot{\tilde{x}}=\tilde{A}\tilde{x}+\tilde{B}u+\tilde{v}$$
$$\tilde{y}=\tilde{C}\tilde{x}+\tilde{D}u$$

where $\tilde{A}=TAT^{-1}$, $\tilde{B}=TB$, $\tilde{C}=CT^{-1}$, and $\tilde{D}$ as earlier defined. The vector $\tilde{v}$ is of course given by $\tilde{v}=Tv$. With the obvious partitioning of all vectors and matrices, the equations may be written as

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}B_1\\ B_2\end{bmatrix}u+\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

$$\begin{bmatrix}y_1\\ y_2\\ y_3\end{bmatrix}=\begin{bmatrix}C_{11} & C_{12}\\ 0 & I\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}D_0\\ 0\\ 0\end{bmatrix}u.$$

Note in particular that $y_2=x_2$. The only effect this step has on the actual synthesis is via the transformation of v, together of course with a transformation of the initial conditions for x. Note, however, that $x_2(0)$ is not needed, since it is available directly from the output vector y_2.

C. Reduction of State-Space Dimension

Finally, define the transformations

$$\hat{y}_1=y_1-C_{12}y_2$$
$$\hat{y}_2=\dot{y}_2-A_{22}y_2-v_2$$
$$\hat{v}=v_1+A_{12}y_2$$

and

$$\hat{x}=x_1.$$

The constructive steps implied by steps B and C are shown in Fig. 2. The new state equations are

$$\dot{\hat{x}}=\hat{A}\hat{x}+\hat{B}u+\hat{v}$$
$$\hat{y}=\hat{C}\hat{x}+\hat{D}u$$

where

$$\hat{A}=[A_{11}]\quad \hat{B}=[B_1]$$
$$\hat{C}=\begin{bmatrix}C_{11}\\ A_{21}\end{bmatrix}\quad \hat{D}=\begin{bmatrix}D_0\\ B_2\end{bmatrix}.$$

This completes one cycle of the algorithm. The new state vector has dimension $\hat{n}=n-q$, and the new output vector has dimension $\hat{p}=r+q\leqslant p$.

At this point there are four possibilities.

1) $\operatorname{Rank}\hat{D}=m=\hat{p}$. That is, $\hat{D}$ is square and nonsingular, and we are done.

2) $\operatorname{Rank}\hat{D}<m\leqslant\hat{p}$. In this case a further cycle of the algorithm is called for.

3) $\hat{p}<m$. In this case further cycles of the algorithm can never lead to case 1) and the algorithm will fail to find an inverse.

4) $\operatorname{Rank}\hat{D}=m<\hat{p}$. It is possible to stop at this point, or to carry out a further cycle of the algorithm. [Ultimately, this will lead to the stopping condition 1).]

In either case 1) or 4), an inverse is immediate. Letting E be any matrix such that $E\hat{D}=I$ [of course, $E=\hat{D}^{-1}$ is the only possibility in case 1)] we have

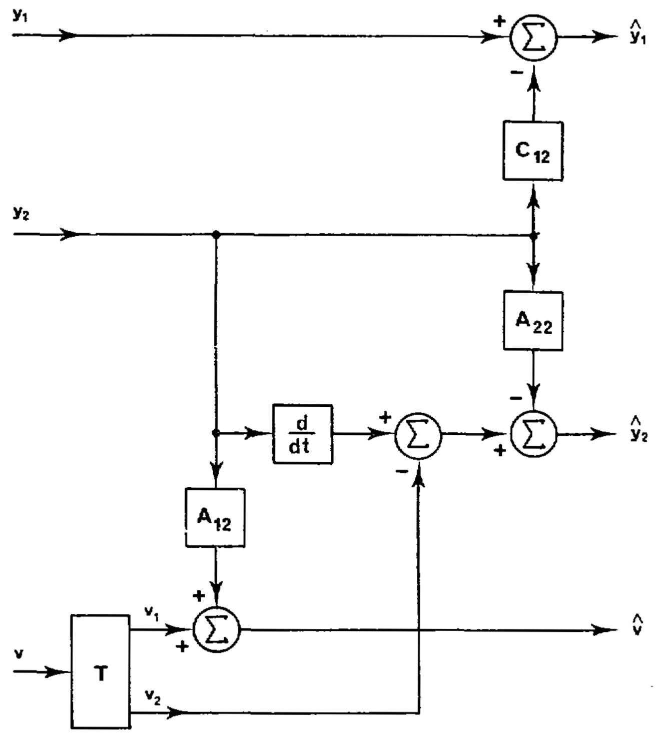

Fig. 2.

$$E\hat{y}=E\hat{C}\hat{x}+u$$

and the inverse system is given by

$$\dot{\hat{x}}=(\hat{A}-\hat{B}E\hat{C})\hat{x}+\hat{B}E\hat{y}+\hat{v}$$
$$u=-E\hat{C}\hat{x}+E\hat{y}.$$

This should of course be concatenated with the partial realizations of Figs. 1 and 2.

It remains to be shown that an inverse will be found whenever one exists. This is not immediate from the construction, since because of the presence of v it is not clear that the transformations of step C are reversable. However, we know from Theorem 1 that an inverse exists only if $\operatorname{rank}M(\lambda)=n+m$ for at least one value of λ. A simple calculation then shows that

$$\operatorname{rank}\hat{M}(\lambda)=\operatorname{rank}\begin{bmatrix}\hat{A}-\lambda I & \hat{B}\\ \hat{C} & \hat{D}\end{bmatrix}=\hat{n}+m$$

with $\hat{A}$, $\hat{B}$, $\hat{C}$, and $\hat{D}$ defined as in step C. This implies that $\hat{p}<m$ is impossible (recall that $\hat{D}$ is $\hat{p}\times m$), so that the algorithm will always find an inverse when one exists. This incidentally completes the proof of Theorem 1.

The sufficiency part of Theorem 2 follows from the observation that values of λ for which $\operatorname{rank}M(\lambda)<n+m$ are precisely those λ for which $\operatorname{rank}\hat{M}(\lambda)<\hat{n}+m$. If $\hat{D}$ is square, these are of course the eigenvalues of $(\hat{A}-\hat{B}\hat{D}^{-1}\hat{C})$ which are in turn the poles of the inverse system. If $\hat{D}$ is not square, then some of the eigenvalues of $(\hat{A}-\hat{B}E\hat{C})$ will depend on E. Relating this back to the stopping conditions for the inversion algorithm, it is found that

1) if the algorithm is halted as soon as possible, i.e., as soon as $\operatorname{rank}\hat{D}=m$, then $\hat{D}$ might not be square and there is no guarantee of stability;

2) if the algorithm is carried out for as many cycles as possible, i.e., $\operatorname{rank}\hat{D}=\hat{p}$ is used as the stopping condition, then $\hat{D}$ will be square and a stable inverse will be found whenever the conditions of Theorem 2 are satisfied.

Note also that there is a tradeoff between the number of differentiators used and the dynamical order (as measured by the number of integrations) of the inverse system. For each state eliminated, a new differentiator is introduced.

Finally, the proof of Theorem 3 may be completed by observing that if $M(\lambda)$ has full rank for *all* λ, then the inversion algorithm will eliminate every state and the inverse will consist only of differentiators. An essentially identical result, obtained, however, by different arguments, was obtained by Bengtsson [17].

IV. Right Inverses

The two definitions below are two "natural" ways in which one might define right invertibility. The first is equivalent, for linear time-invariant systems, to that given by Brockett and Mesarović [1] or that of Doty and Frank [15]. The second appears to be new.

Definition: System (1) is *functionally reproducible* if, for any $y_{\text{ref}}(\cdot)$ defined on $[0,\infty)$ and any $x(0)$, there exists a $u(\cdot)$ such that $y(t)=y_{\text{ref}}(t)$ for all $t\in[0,\infty)$.

Definition: System (1) is *right invertible* if, for any $y_{\text{ref}}(\cdot)$ defined on $[0,\infty)$, there exists a $u(\cdot)$ and a choice of $x(0)$ such that $y(t)=y_{\text{ref}}(t)$ for all $t\in[0,\infty)$.

The only difference between the definitions is in the way the initial state is handled. (In both definitions y_{ref} is of course constrained to have the differentiability properties imposed in Section II.) It turns out, though, that our system is functionally reproducible only if $\operatorname{rank}(D)=p$; for all other cases, it is impossible to specify $x(0)$ and $y(0)$ independently. We choose therefore to work from the second definition.

The conditions for right invertibility will now be stated. In the statement of Theorem 4 a right inverse will be said to be "stable" if it is asymptotically stable in the sense of Lyapunov. It will be said to be a "right inverse with unknown initial state" if the initial state of the original system (1) is irrelevant in constructing the inverse.

Theorem 4: System (1) possesses a right inverse if and only if

$$\operatorname{rank} M(\lambda)=n+p$$

for some real λ. It possesses a stable right inverse if and only if the condition is satisfied for all complex λ in $\operatorname{Re}(\lambda)\geqslant 0$, and a right inverse with unknown initial state if and only if the condition holds for all complex λ.

A proof may be found in [28]. The actual construction of the right inverse proceeds via a trivial modification of the algorithm of Section III.

Finally, we can establish a connection between the present invertibility definitions and a frequency-domain formulation of the problem.

Theorem 5: The system (1) is left (right) invertible if and only if the transfer function matrix

$$Z(s)=D+C(sI-A)^{-1}B$$

has linearly independent columns (rows) for all but a finite number of complex s.

Proof: Following Dorato [5] we have the identity

$$\begin{bmatrix} A-sI & B\\ C & C\end{bmatrix}\begin{bmatrix} I & (sI-A)^{-1}B\\ 0 & I\end{bmatrix}=\begin{bmatrix} A-sI & 0\\ C & Z(s)\end{bmatrix}.$$

Combining this with Theorems 1 and 4 the result follows immediately. ∇∇∇

V. Finite State Systems

The arguments of the preceding sections carry over almost without change to cover the inversion of the discrete-time system

$$\begin{aligned} x(k+1)&=Ax(k)+Bu(k)\\ y(k)&=Cx(k)+Du(k). \end{aligned} \tag{3}$$

The only changes required are substitution of λ^k everywhere for $e^{\lambda t}$, and consequently $|\lambda|\geqslant 1$ for $\operatorname{Re}(\lambda)\geqslant 0$ in the stability criteria. The change required in the inversion algorithm is the substitution of "unit predictors" for differentiators; this makes the inverse noncausal, but with the obvious change to the time scale a causal inverse with delay is produced.

More generally, many of the arguments remain valid when all matrices and vectors have elements in an arbitrary field F. An important special case occurs when F is a finite field, since then (3) represent a class of linear finite automata. The problem of inverting such a system is equivalent to the problem of decoding a discrete convolutional code [3], [25], [26].

Note that while the inversion algorithm applies regardless of the nature of the field F, Theorems 1–4 make use of complex numbers and hence are not valid generally (apart from their intrinsic interest, these theorems are needed to guarantee that, for example, the inversion algorithm will find a stable inverse whenever one exists). When F is a finite field, this situation may be rectified as follows.

Recalling that n is the dimension of the state vector, let F_e be a finite extension of F containing the roots of all nth degree polynomials (that is, F_e is a field containing all of the elements of F together perhaps with some other elements. The existence of such extensions is well known [27]). Of course, F_e is a finite field; more importantly, it is a vector space over F [27]. That is, there exists a set $S=\{e_i\}$—not, in general, unique—of elements of F_e such that every element α of F_e can be written in the form

$$\alpha=\sum \alpha_i e_i$$

with coefficients α_i in F. There is of course no loss of generality in choosing $e_0=1$.

Before proceeding further, we need to define stability for a finite-state system.

Definition: The finite-state system (5) is stable if and only if whenever $u(k)=0$ for all k, and for any $x(0)$, there exists a $k_0\geqslant 0$ such that $x(k)=0$ for all $k>k_0$.

It is readily verified from this definition (and assuming complete controllability and observability) that the following five statements are equivalent.

1) The system (3) is stable.
2) For some $k_0\geqslant 0$, $A^{k_0}=0$.
3) There exists a feedback-free realization of (3).
4) If $u(k)$ is nonzero for only a finite number of k, then so is $y(k)$.
5) $\det(A-\lambda I)$ is nonzero for all $\lambda\neq 0$ in F_e.

Statements 1) and 4) are of greatest interest to coding theorists—in particular, 4) is a statement about finite error propagation—while 5) is more useful in the present context. For further details, see [3], [5], [16], [25], and [26]. Of course, the issue here is not so much the stability of (5) as the stability of its inverse. This only requires that A be replaced above by the appropriate matrix determined by the inversion algorithm.

With these preliminaries over, we have the following result.

Theorem 6: Suppose all vectors and matrices in (3) have components in a finite field F. Let F_e be the extension field defined earlier. Then the system (3) is left (right) invertible if and only if

$$M(\lambda)\triangleq\begin{bmatrix} A-\lambda I & B\\ C & D\end{bmatrix}$$

has linearly independent columns (rows) for at least one $\lambda\in F$. It has a stable left (right) inverse if and only if $M(\lambda)$ has linearly independent columns (rows) for all $\lambda\neq 0$ in F_e, and is left (right) invertible with unknown initial state if and only if these conditions hold for all $\lambda\in F_e$.

Proof: With the basis set S defined as in the preamble to the theorem, let $P_S(\cdot)$ be the projection of F_e onto F defined via

$$P_S\left(\sum \alpha_i e_i\right)=\alpha_0.$$

Then the proofs of Theorems 1–4 are valid in the present context if $\operatorname{Re}(\cdot)$ is replaced everywhere by $P_S(\cdot)$ and $e^{\lambda t}$ is replaced everywhere by λ^k. ∇∇∇

Of course, the conditions of Theorem 6 involve operations over F_e and so might not be easy to check. However, this point is unimportant in view of the obvious corollary: the system (3) has an inverse with the desired properties if and only if the inversion algorithm finds such an inverse.

VI. Conclusions

A new inversion algorithm has been presented for the left or right inversion of a linear system. It is computationally efficient since at each step the dimensionality of the problem is reduced, and because the operations within each step consist simply of elementary row or column operations on a set of matrices. At the same time new invertibility criteria have been presented which serve both to justify the algorithm and to delineate in a simple way the class of invertible and stably invertible systems.

Applications have been mentioned only briefly, but numerous applications may be found in the references cited. These include, but are not restricted to, solutions to decoupling problems, new approaches to the design of controllers, and decoding of convolutional codes.

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Controllability of a Class of Discrete Time Bilinear Systems

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Abstract—**In this paper the controllability of a class of discrete time bilinear systems is discussed. Necessary and sufficient conditions for complete controllability are derived. In addition a complete characterization of the controllable regions of this class of systems, when not completely controllable, is made.**

I. Introduction

The study of bilinear systems has received considerable attention in the last few years. Such systems have been successfully used to model a variety of physical phenomena for which linear model representation has proved inadequate [1]–[3].

In this paper we study the controllability of homogeneous discrete time bilinear systems

$$x_{k+1}=[A+u_k B]x_k, \qquad k=0,1,\cdots \tag{1}$$

with scalar control sequence $\{u_k,\ k=0,1,\cdots\}$.

In [4] and [5] necessary (but not sufficient) conditions and sufficient (but not necessary) conditions for the controllability of (1) have been given when the matrices A and B are $n\times n$ constant real matrices and B has rank 1. We shall completely resolve the issue for this problem by giving (Theorem 2) an easily checked set of conditions which are both necessary and sufficient for the controllability of (1) (rank $B=1$). An example from [4] is discussed to illustrate our results.

In addition, we examine the structure of systems which are not completely controllable and give a complete characterization of the regions of controllability.

Results for controllability with bounded controls can be found in [7].

II. Homogeneous Bilinear System

It has been shown [4] that we need consider nonsingular A only. Since rank $B=1$ and A is nonsingular, it follows that rank $A^{-1}B=1$ and hence

$$A^{-1}B=ch^T \tag{2}$$

where c and h are constant n-vectors of appropriate dimensions. It is readily verified that for an arbitrary number of steps r, from an initial state x_0, x_r can be expressed as

$$x_r=A^r\prod_{j=0}^{r-1}[I+D_j]x_0 \tag{3}$$

where

$$\prod_{j=0}^{r-1}[I+D_j]=[I+D_{r-1}][I+D_{r-2}]\cdots[I+D_0]$$

with

$$D_j=u_jA^{-j}ch^TA^j.$$

III. A Modified System (The Companion System)

Consider the system

$$y_r=\prod_{j=0}^{r-1}[I+D_j]x_0 \tag{4}$$

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