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THE BRUNE SYNTHESIS IN STATE–SPACE TERMS*

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SUMMARY

A state-space interpretation of the one-port Brune synthesis of a rational positive real function is presented. A natural generalization then leads to a multiport synthesis for rational positive real matrices.

INTRODUCTION

The development of synthesis procedures for passing from a prescribed rational positive real function or matrix to a linear lumped network of passive components possessing the prescribed quantity as its impedance has been one of the major problems confronting network theorists in the past. One of the earliest such synthesis procedures is the Brune synthesis, see Reference 1 and for more recent treatments, References 2 and 3.

As presented in References 1–3, the synthesis is of positive real functions, rather than positive real matrices. Multiport generalizations may also be found.[4–9] Reference 9 by Newcomb, besides referencing earlier technical reports by the same author, provides comparisons of the various multiport approaches.

Our goal here is twofold. First, we aim to present the Brune cycle in state-space terms. As we show here, carrying out a Brune cycle is equivalent to the problem of finding a state-space description of an impedance in a special co-ordinate basis. In this special co-ordinate basis, a certain matrix appearing in the so-called Positive Real Lemma[10–13]—the fundamental result on positive realness in state-space terms—takes on a special form which enables its part identification. Since, as argued in Reference 10, synthesis in some ways is equivalent to the complete identification of this matrix, it becomes reasonable that part identification corresponds to part synthesis.

Our second aim is to illustrate how the multiport case is a natural extension of the single port case when viewed in state-space terms—perhaps more so than when a classical viewpoint is taken. Further, and as one would hope, in synthesizing a symmetric impedance matrix, gyrators are automatically excluded.

The layout of the paper is as follows. In section 2 we analyse, as opposed to synthesize, the structure resulting from a Brune cycle, to exhibit the sort of state-space equations one needs in order to execute a synthesis step. Section 3 contains our fundamental lemma, explaining how one can change the co-ordinate basis to get the right equations (The proof has common roots with that used by Yakubovic in a proof of the Positive Real Lemma[11].) The section 'One-Port Brune Synthesis' explains the one-port synthesis, and the section 'Multiport Brune Synthesis' with the aid of a generalization of the fundamental lemma, discusses the multiport problem.

We caution the reader that in the course of the paper, some familiarity with the Brune synthesis is expected, including some familiarity with the properties of positive real functions and matrices. References 2, 3, 9 will provide adequate background. Further, some familiarity with state-variable descriptions of networks is also expected, see e.g. Reference 10.

PRELIMINARY ANALYSIS

We shall study the arrangement of Figure 1, and show how a state-space description of the network N_1 is related to a state-space description of N. This will enable presentation of the fundamental lemma relevant to the synthesis problem in the next section.

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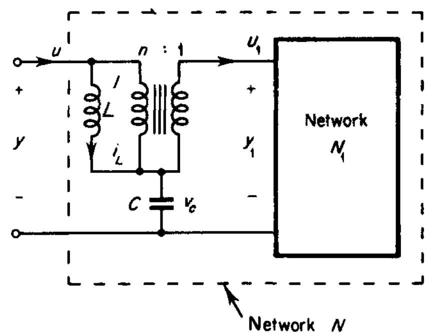

Figure 1. A network N comprising a Brune section and a terminating network N_1

Suppose that N_1 is described by state-space equations

$$\dot{\mathbf{x}}_1 = \mathbf{F}_1\mathbf{x}_1+\mathbf{G}_1\mathbf{u}_1 \qquad \mathbf{y}_1 = \mathbf{H}_1'\mathbf{x}_1+\mathbf{J}_1\mathbf{u}_1 \tag{1}$$

Here, x_1 is *not* necessarily a vector of capacitor voltages and inductor currents. It may simply be the state-vector of a state-variable equation set derived from the impedance function of N_1, and no more.

With quantities as defined in Figure 1, it is evident that a state vector for N could be taken to be $[x_1' \quad x_2 = \sqrt{C}v_c \quad x_3 = \sqrt{L}i_L]'$. With these definitions, the state-variable equations of N turn out after some manipulation to be

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} F_1 & 0 & -\dfrac{n}{\sqrt{L}}G_1 \\ 0 & 0 & \dfrac{n}{\sqrt{LC}} \\ \dfrac{nH_1'}{\sqrt{L}} & -\dfrac{n}{\sqrt{LC}} & -\dfrac{n^2J_1}{L} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} nG_1 \\ \dfrac{1-n}{\sqrt{C}} \\ \dfrac{n^2J_1}{\sqrt{L}} \end{bmatrix} u \\ y &= \begin{bmatrix} nH_1' & \dfrac{1-n}{\sqrt{C}} & -\dfrac{n^2J_1}{\sqrt{L}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + n^2J_1u \end{aligned} \tag{2}$$

Several important points should be noted. First, suppose that $\dot{\mathbf{x}} = \mathbf{F}_a\mathbf{x}+\mathbf{G}_a\mathbf{u}$, $\mathbf{y} = \mathbf{H}_a'\mathbf{x}+\mathbf{J}\mathbf{u}$ is a state-space set of equations for N, in the sense that $z(s) = \mathbf{J}+\mathbf{H}_a'(s\mathbf{I}-\mathbf{F}_a)^{-1}\mathbf{G}_a$ is the impedance function of N. Suppose also that by change of co-ordinate basis, these equations could be carried into the form (2), for some $\mathbf{F}_1$, $\mathbf{G}_1$, $\mathbf{H}_1$, $\mathbf{J}_1$, n, $\mathbf{L}$ and $\mathbf{C}$. Then the problem of synthesizing N becomes one of synthesizing N_1 with N_1 defined by $\dot{\mathbf{x}}_1 = \mathbf{F}_1\mathbf{x}_1+\mathbf{G}_1\mathbf{u}$, $\mathbf{y} = \mathbf{H}_1'\mathbf{x}+\mathbf{J}_1\mathbf{u}$.

Of course, it may not be true that an arbitrary state-variable realization can be converted to one of the form (2) and this brings us to our next point: If (2) holds, and if $z(s)$ is the impedance function linking u and y in (2), then $z(j\omega_0)+z(-j\omega_0) = 0$ for $\omega_0 = \sqrt{(n/LC)}$. (Those familiar with the Brune synthesis will recognize this property immediately. It may also be proved directly from (2).) In relation to the first point made, this means that *before* one can think of transforming an arbitrary state-variable equation set to one of the form (2), it must be the case that for some real ω_0, $z(j\omega_0)+z(-j\omega_0) = 0$.

The third point to be made is tied up with the so-called Positive Real Lemma.[10–13] If $\{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$ is a quadruple of matrices defining a minimal state-variable realization of a square $m\times m$ real rational matrix $\mathbf{Z}(s)$ with $Z(\infty) < \infty$, the Lemma states that $\mathbf{Z}(s)$ is positive real if and only if there exists a positive definite $\mathbf{P}$ and matrices $\mathbf{M}$ and $\mathbf{V}$ such that

$$\mathbf{PF}+\mathbf{F'P} = -\mathbf{MM'} \qquad \mathbf{PG} = \mathbf{H}-\mathbf{MV} \qquad \mathbf{V'V} = \mathbf{J}+\mathbf{J'} \tag{3}$$

Almost all state-variable passive synthesis techniques for passing from a prescribed state-variable realization of a positive real $\mathbf{Z}(s)$ to a network synthesizing it have as their key step the determination of a matrix $\mathbf{P}$ satisfying (3). Turning to our particular problem, suppose $\mathbf{P}_1$, $\mathbf{M}_1$ and $\mathbf{V}_1$ are matrices for which $\mathbf{P}_1\mathbf{F}_1 + \mathbf{F}'_1\mathbf{P}_1 = -\mathbf{M}_1\mathbf{M}'_1$ and so on. Then with $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{J}$ defined as the global matrices appearing in (2), it is not hard to check that

$$\mathbf{P} = \mathbf{P}_1 \oplus \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \mathbf{M}'_1 & 0 & -\dfrac{n\mathbf{V}_1}{\sqrt{L}} \end{bmatrix}' \qquad \mathbf{V} = n\mathbf{V}_1 \tag{4}$$

causes satisfaction of (3). In other words, knowledge of a solution of the Positive Real Lemma equation associated with the realization $\{\mathbf{F}_1, \mathbf{G}_1, \mathbf{H}_1, \mathbf{J}_1\}$ of N_1 yields knowledge of a solution of the equation associated with the realization $\{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$ of N.

One can turn this idea round as follows: Suppose a positive real $\mathbf{Z}(s)$ has a minimal realization $\{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$ of the form of (2); then the task of finding a $\mathbf{P}$ satisfying (3) is reduced. Part of $\mathbf{P}$ is identifiable immediately as a 2×2 identity matrix. The remaining part of $\mathbf{P}$ is defined as any positive definite $\mathbf{P}_1$ causing satisfaction of (3) with the subscript 1 on every matrix. Put another way, determination of a co-ordinate basis change taking an arbitrary realization $\{\mathbf{F}_a, \mathbf{G}_a, \mathbf{H}_a, \mathbf{J}\}$ of $\mathbf{Z}(s)$ to the form (2) partly identifies $\mathbf{P}$ at the same time, and in this sense can be thought of as moving towards a solution of the synthesis problem.

In a sense, the decomposition of $\mathbf{P}$ follows on physical grounds. As discussed in Reference 10, the quadratic form $\frac{1}{2}\mathbf{x}'\mathbf{P}\mathbf{x}$ can be thought of as a stored energy for the network. The definitions of $\mathbf{x}_2$ and $\mathbf{x}_3$ and study of Figure 1 then show that if the stored energy of N_1 is $\frac{1}{2}\mathbf{x}'_1\mathbf{P}_1\mathbf{x}_1$, that of N will be $\frac{1}{2}[\mathbf{x}'_1\mathbf{P}_1\mathbf{x}_1 + \mathbf{x}'_2\mathbf{x}_2 + \mathbf{x}'_3\mathbf{x}_3]$, i.e. $\frac{1}{2}\mathbf{x}'\mathbf{P}\mathbf{x}$ where $\mathbf{P}_1$ and $\mathbf{P}$ are related as in (4).

THE FUNDAMENTAL LEMMA

In this section we shall show that the condition $z(j\omega_0) + z(-j\omega_0) = 0$, which is necessary for the existence of a state-variable description of the type (2), is also sufficient for there to be such a description. This will enable statement of the Brune synthesis in the next section.

Lemma

One-port case. Let $\{\mathbf{F}_a, \mathbf{G}_a, \mathbf{H}_a, \mathbf{J}\}$ be a minimal realization of a positive real impedance function $z(s)$ such that $z(j\omega_0) + z(-j\omega_0) = 0$ for some real finite and nonzero ω_0 with $j\omega_0$ not an eigenvalue of $\mathbf{F}_a$. Suppose also that $\mathbf{F}_a$ is of dimension greater than 1×1. Then there exists a co-ordinate basis change matrix $\mathbf{T}$ such that with $\mathbf{F} = \mathbf{T}\mathbf{F}_a\mathbf{T}^{-1}$, etc., the quadruple $\{\mathbf{F}, \mathbf{G}, \mathbf{H}, \mathbf{J}\}$ has the form as shown in (2). Further, $\mathbf{T}$ may be computed from $\{\mathbf{F}_a, \mathbf{G}_a, \mathbf{H}_a, \mathbf{J}\}$ and ω_0 in closed-form.

Outline proof. A sequence of basis change matrices can be defined as follows:

(i) Let $\mathbf{T}_a$ be any nonsingular matrix for which the last two columns of $\mathbf{T}_a^{-1}$ are $(\omega_0^2\mathbf{I} + \mathbf{F}_a^2)^{-1}\mathbf{G}_a$ and $-\mathbf{F}_a(\omega_0^2\mathbf{I} + \mathbf{F}_a^2)^{-1}\mathbf{G}_a$;

(ii) Setting $\mathbf{F}_b = \mathbf{T}_a\mathbf{F}_a\mathbf{T}_a^{-1}$ and $\mathbf{H}_b = (\mathbf{T}_a^{-1})'\mathbf{H}_a$, compute

$$[(\omega_0^2\mathbf{I} + \mathbf{F}_b'^2)^{-1}\mathbf{H}_b \qquad \mathbf{F}'_b(\omega_0^2\mathbf{I} + \mathbf{F}_b'^2)^{-1}\mathbf{H}_b] = \begin{bmatrix} \mathbf{K}_{12} \\ \mathbf{K}_{22} \end{bmatrix}$$

Then set

$$\mathbf{T}_b = \begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{K}_{22}^{-1}\mathbf{K}'_{22} & \mathbf{I} \end{bmatrix}$$

(iii) Set $\mathbf{F}_c = \mathbf{T}_b\mathbf{F}_b\mathbf{T}_b^{-1}$ and $\mathbf{H}_c = (\mathbf{T}_b^{-1})'\mathbf{H}_b$. Then it turns out that

$$[(\omega_0^2\mathbf{I} + \mathbf{F}_c'^2)^{-1}\mathbf{H}_c \qquad \mathbf{F}'_c(\omega_0^2\mathbf{I} + \mathbf{F}_c'^2)^{-1}\mathbf{H}_c] = \begin{bmatrix} 0 & 0 & \dots & \alpha^2 & 0 \\ 0 & 0 & \dots & 0 & \beta^2 \end{bmatrix}$$

for nonzero real scalars α and β. Define

$$\mathbf{T}_c = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

Finally, let $\mathbf{T} = \mathbf{T}_c\mathbf{T}_b\mathbf{T}_a$ and define $\mathbf{F} = \mathbf{T}\mathbf{F}_a\mathbf{T}^{-1}$, $\mathbf{G} = \mathbf{T}\mathbf{G}_a$ and $\mathbf{H} = (\mathbf{T}^{-1})'\mathbf{H}_a$. Then it turns out that $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{J}$ are of the required form. The proof of all the above assertions rests on straightforward applications of the Positive Real Lemma.

ONE-PORT BRUNE SYNTHESIS

With the fundamental lemma of the previous section in hand, we can now describe how a Brune synthesis of a positive real $z(s)$ is carried out.

As usual, the synthesis is carried out *via* a sequence of cycles. The first task in each cycle is, again as usual, to remove $j\omega$-axis poles and zeros *via* the 'Foster preamble' to the Brune synthesis, see e.g. References 2 and 3. The impedance remaining to be synthesized after the preamble, call it $z(s)$, is then finite and nonzero at $s = \infty$.

The next step of any one Brune cycle is to find the value of real ω where $\mathrm{Re}\,[z(j\omega)]$ takes its minimum; if the minimizing ω is 0 or ∞, the situation is easily dealt with, and will not be discussed.

If the minimizing ω is nonzero and finite, say $\omega = \omega_0$, set $r_0 = \mathrm{Re}\,[z(j\omega_0)]$; then $\hat{z}(s) = z(s) - r_0$ is positive real, has $\hat{z}(j\omega_0) + \hat{z}(-j\omega_0) = 0$, and a synthesis of $z(s)$ follows from series connections of a resistor r_0 and a synthesis of $\hat{z}(s)$. Now the fundamental lemma can be applied to a minimal state-variable realization of $\hat{z}(s)$, to complete one Brune cycle.

As usual, each Brune cycle reduces the degree of the impedance remaining to be synthesized at the end of the cycle, and so the process eventually terminates.

The hard part is of course determination of the ω for which $\mathrm{Re}\,[z(j\omega)]$ is minimum. Below, we suggest three techniques which may be used for this step.

(i) Forming $\mathrm{Re}\,[z(j\omega)]$ as a ratio of two even polynomials in ω, standard minimization techniques may be used to locate the minimum. (This would normally be done by locating the zeros of $\partial/\partial\omega\,\mathrm{Re}\,[z(j\omega)]$.) Conceptually, this technique is the simplest, but it may be numerically ill-conditioned because of repeated stationary points in $\mathrm{Re}\,[z(j\omega)]$.

(ii) As shown in Reference 10, the Riccati equation

$$\dot{\mathbf{P}} = \mathbf{P}(\mathbf{F} - \mathbf{G}\mathbf{R}^{-1}\mathbf{H}') + (\mathbf{F}' - \mathbf{H}\mathbf{R}^{-1}\mathbf{G}')\mathbf{P} - \mathbf{P}\mathbf{G}\mathbf{R}^{-1}\mathbf{G}'\mathbf{P} + \mathbf{H}\mathbf{R}^{-1}\mathbf{H}' \tag{5}$$

with initial condition $\mathbf{P}(0) = \mathbf{0}$ has a finite escape time if and only if $\bar{z}(s) = \frac{1}{2}\mathbf{R} + \mathbf{H}'(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$ is not positive real. Now if $z(s) = \mathbf{J} + \mathbf{H}'(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$, one can solve (5) for various real positive scalars R to find that $\hat{R}$ which separates the solutions with finite escape time from those without. Then it will be true that $\hat{z}(j\omega) = z(j\omega) - (J - \frac{1}{2}\hat{R})$ will have zero real part for some ω, which may then be determined.

(iii) Again with $z(s) = \mathbf{J} + \mathbf{H}'(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$, consider the matrix

$$\mathbf{X}(R) = \begin{bmatrix} \mathbf{F} - \mathbf{G}\mathbf{R}^{-1}\mathbf{H}' & -\mathbf{G}\mathbf{R}^{-1}\mathbf{G}' \\ \mathbf{H}\mathbf{R}^{-1}\mathbf{H}' & -\mathbf{F}' + \mathbf{H}\mathbf{R}^{-1}\mathbf{G}' \end{bmatrix}$$

It is not hard to show that the eigenvalues of $\mathbf{X}(R)$ are the zeros of $\bar{z}(s) + \bar{z}(-s)$, where $\bar{z}(s) = \frac{1}{2}\mathbf{R} + \mathbf{H}'(s\mathbf{I} - \mathbf{F})^{-1}\mathbf{G}$. If R varies from some large positive number towards zero, $\mathbf{X}(R)$ will initially have no pure imaginary eigenvalues of even order, but such eigenvalues will appear as R is reduced. Suppose that an eigenvalue $j\omega_0$ first appears for $R = \hat{R}$. Then it is easily seen that $\min_\omega \mathrm{Re}\,[z(j\omega)] = J - \frac{1}{2}\hat{R}$, with the minimum actually occurring at $\omega = \omega_0$.

MULTIPORT BRUNE SYNTHESIS

In this section, the earlier ideas are applied to the multiport problem. First, consider the arrangement depicted in Figure 2; several comments are in order. Now $\bar{N}$ is a multiport network. One of the port currents is u_A and the remainder form a vector $\mathbf{u}_B$. A type of Brune section when cascaded with a network N_1 yields $\bar{N}$. The section varies from that of Figure 1 by the inclusion of the transformers with turn-ratios $1:v_k$. Though only one transformer is shown, a number are to be understood as being used. The primaries are all in parallel. For the secondaries, it is understood that the terminals marked k are interconnected, as are the terminals marked k', and the kth entry of $\mathbf{u}_B$ defines the current flowing through $k'-k$. The vector $[v_1 \ v_2 \ \ldots \ ]'$ will be denoted by $\mathbf{v}$.

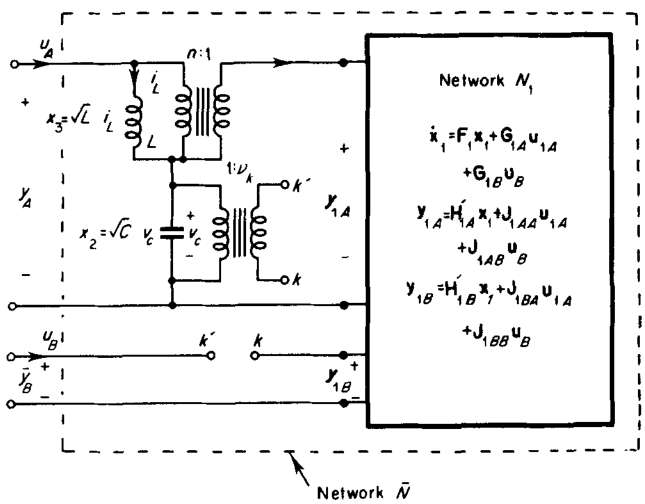

Figure 2. A network $\bar{N}$ showing the main modification in the Brune section

By inspection, it is found that $\bar{N}$ has state equations

$$
\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_1 & 0 & -\dfrac{n\mathbf{G}_{1A}}{\sqrt{L}} \\ 0 & 0 & \dfrac{n}{\sqrt{LC}} \\ \dfrac{n\mathbf{H}_{1A}}{\sqrt{L}} & -\dfrac{n}{\sqrt{LC}} & -\dfrac{n^2\mathbf{J}_{1AA}}{\sqrt{L}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} n\mathbf{G}_{1A} & \mathbf{G}_{1B} \\ \dfrac{1-n}{\sqrt{C}} & \dfrac{\mathbf{v}'}{\sqrt{C}} \\ \dfrac{n^2\mathbf{J}_{1AA}}{\sqrt{L}} & \dfrac{n\mathbf{J}_{1AB}}{\sqrt{L}} \end{bmatrix} \begin{bmatrix} u_A \\ \mathbf{u}_B \end{bmatrix}
$$

$$
\begin{bmatrix} \bar{y}_A \\ \bar{\mathbf{y}}_B \end{bmatrix} = \begin{bmatrix} n\mathbf{H}_{1A} & \dfrac{1-n}{\sqrt{C}} & -\dfrac{n^2\mathbf{J}_{1AA}}{\sqrt{L}} \\ \mathbf{H}'_{1B} & \dfrac{\mathbf{v}}{\sqrt{L}} & -\dfrac{n\mathbf{J}_{1BA}}{\sqrt{L}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} n^2\mathbf{J}_{1AA} & n\mathbf{J}_{1AB} \\ n\mathbf{J}_{1BA} & \mathbf{J}_{1BB} \end{bmatrix} \begin{bmatrix} u_A \\ \mathbf{u}_B \end{bmatrix} \tag{6}
$$

Now consider the arrangement of Figure 3. The gyrator shown in the figure is shorthand for a number of gyrators. One side of each gyrator appears in one and only one of the input lines of the 'B set' of ports; the other sides of all the gyrators are connected in series, in the input line of port A. The effect is that

$$\mathbf{y}_B - \bar{\mathbf{y}}_B = \gamma u_A$$
$$y_A - \bar{y}_A = -\gamma'\mathbf{u}_B$$

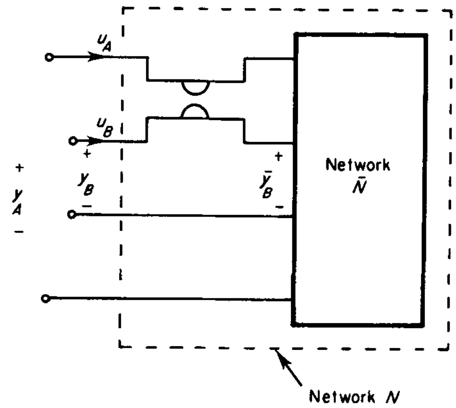

Figure 3. The network N as a series connection of gyrators and the network $\bar{N}$ of Figure 2. Together, Figures 2 and 3 show the Brune section

where γ is the vector of gyrator impedances. The state-space equations of N differ but little from those of $\bar{N}$. Equation (6) is unaltered, while (7) is replaced by

$$\begin{bmatrix} y_A \\ \mathbf{y}_B \end{bmatrix} = \begin{bmatrix} n\mathbf{H}'_{1A} \\ \mathbf{H}'_{1B} \end{bmatrix} \begin{bmatrix} \dfrac{1-n}{\sqrt{C}} & -\dfrac{n^2\mathbf{J}_{1AA}}{\sqrt{L}} \\ \dfrac{\mathbf{v}}{\sqrt{C}} & -\dfrac{n\mathbf{J}_{1BA}}{\sqrt{L}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} n^2\mathbf{J}_{1AA} & n\mathbf{J}_{1AB} \\ n\mathbf{J}_{1BA} & \mathbf{J}_{1BB} \end{bmatrix} \begin{bmatrix} u_A \\ \mathbf{u}_B \end{bmatrix}$$

The reader should now appreciate that if, given an arbitrary minimal realization of positive impedance matrix $\mathbf{Z}(s)$, say $\{\mathbf{F}_a, \mathbf{G}_a, \mathbf{H}_a, \mathbf{J}\}$, we can find a co-ordinate basis transformation giving a new realization of the form in (6) and (8), then, in effect, a Brune cycle can be completed. The synthesis problem is then reduced to one of synthesizing the impedance matrix with realization $\{\mathbf{F}_1, \mathbf{G}_1, \mathbf{H}_1, \mathbf{J}_1\}$, where $\mathbf{G}_1 = [\mathbf{G}_{1A} \;\; \mathbf{G}_{1B}]$, etc.

As in the one-port case, such a transformation cannot necessarily be found. This is because (6) and (8) imply that $Z_{11}(j\omega_0)+Z_{11}(-j\omega_0) = 0$ where $\omega_0 = \sqrt{(n/LC)}$, and $Z_{11}(s)$ is the 1–1 entry of $\mathbf{Z}(s)$. The content of the Fundamental Lemma is that if this condition holds, then one can construct the transformation.

In presenting the Lemma, we shall rule out slightly more $\mathbf{Z}(s)$ than in the one-port case. As before, we shall require $\mathbf{F}_a$ to have dimension greater than 1×1 and no eigenvalue at $j\omega_0$ (If $\mathbf{F}_a$ is 1×1, then synthesis is easy and does not need a Brune section. If $\mathbf{F}_a$ has an eigenvalue at $j\omega_0$ extraction of a multiport Foster Section will eliminate it[9].) We shall also suppose $\mathbf{F}_a$ is nonsingular (a zero eigenvalue can be removed, again through extraction of a Foster section). Finally, we shall rule out the possibility of having

$$\mathbf{Z}(s) = \begin{bmatrix} Z_{11}(s) & Z_{12}(s) \\ Z_{21}(s) & Z_{22}(s) \end{bmatrix} = \begin{pmatrix} 0 & \mathbf{\Gamma}' \\ -\mathbf{\Gamma} & Z_{22}(s) \end{pmatrix} \tag{9}$$

for some real constant vector $\mathbf{\Gamma}$. If this were the case, synthesis of $\mathbf{Z}(s)$ would of course follow easily from synthesis of $Z_{22}(s)$.

Lemma

Multiport case. Let $\{\mathbf{F}_a, \mathbf{G}_a, \mathbf{H}_a, \mathbf{J}\}$ be a minimal realization of a positive real impedance matrix $\mathbf{Z}(s)$ such that $Z_{11}(j\omega_0)+Z_{11}(-j\omega_0) = 0$ for some finite and nonzero ω_0. Suppose that $\mathbf{F}_a$ is of dimension greater than 1×1, is nonsingular, and $j\omega_0$ is not an eigenvalue. Suppose also that $\mathbf{Z}(s)$ does not have the form shown in (9). Then there exists a co-ordinate basis change matrix $\mathbf{T}$, computable from $\{\mathbf{F}_a, \mathbf{G}_a, \mathbf{H}_a, \mathbf{J}\}$ and ω_0 in closed form, such that with $\mathbf{F} = \mathbf{T}\mathbf{F}_a\mathbf{T}^{-1}$, etc., the state-space equations $\dot{\mathbf{x}} = \mathbf{F}\mathbf{x}+\mathbf{G}\mathbf{u}$, $\mathbf{y} = \mathbf{H}'\mathbf{x}+\mathbf{J}\mathbf{u}$ have the form depicted in (6) and (8).

Outline proof†: Let $\mathbf{e}_1$ be a vector of all zeros, except for 1 in the last entry. Replacing $\mathbf{G}$ and $\mathbf{H}$ by $\mathbf{Ge}_1$ and $\mathbf{He}_1$, apply the construction for the one-port case. Then $\mathbf{F}$, $\mathbf{G}$, $\mathbf{H}$ and $\mathbf{J}$ turn out to be precisely of the required form. The computations are identical to those for the one-port case; their justification is slightly different, but still relies solely on the Positive Real Lemma.

What now does the full Brune synthesis procedure involve? We refer the reader to Reference 9 for details. Each cycle commences with a multiport Foster preamble. Second, one carries out a resistor extraction at one port, and follows this with a transformer extraction. By selecting the resistance and transformer appropriately, the impedance remaining for synthesis has $1-1$ term with a real part of zero at $s = j\omega_0$ for some real ω_0. If ω_0 is nonzero and finite, the preceding analysis applies, while if ω_0 is zero or infinite, straightforward modifications of the one-port case again apply.

The reader will have observed the occurrence of gyrators in the section removed in the basic cycle. The vector of gyration impedances is $\boldsymbol{\gamma}$, and appears in equation (8). One might well ask whether these gyrators are essential if the original impedance $\mathbf{Z}(s)$ is symmetric. The answer is no. In fact, with $Z_{12}(s)$ and $Z_{21}(s)$ as defined in (9), equating of $Z_{12}(j\omega_0)$ and $Z'_{21}(j\omega_0)$ shows that $\boldsymbol{\gamma} = 0$. In other words, symmetry of $\mathbf{Z}(s)$ [in fact, symmetry only of $\mathbf{Z}(j\omega_0)$] implies that no gyrators will appear in the section extracted.

CONCLUSIONS

We have described the one-port Brune synthesis using state-space ideas, and then exhibited a multiport extension which is a natural and fairly straightforward generalization of the one-port case. Further, the generalization handles symmetric and nonsymmetric $\mathbf{Z}(s)$ with equal ease, ensuring in the symmetric case that no gyrators occur in the section removed.

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† A detailed proof may be found in a technical report obtainable from the authors.