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Implications of Passivity in a Class of Nonlinear Systems

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Abstract—For a broad class of nonlinear systems, a connection is established between the input-output property of passivity and a set of constraints on the state equations of the system. These constraints are then interpreted in terms of the stored energy and dissipation of a passive system. Applications are given in two problems of optimal control theory, and a generalized form of the circle criterion is also derived.

I. Introduction

In electrical network theory, the distinction between active and passive networks has long been considered an important one. The distinction is simple; active networks may contain internal energy sources, whereas passive networks—at least in principle—do not. Because of the obvious physical implications of classifying networks in this way, a large body of literature has grown up on the subject of passive networks [1],[2].

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For more general control systems, the physical meaning and implications of passivity are less clear. The usual approach—and one that will be adopted in this paper—is to proceed by analogy with the networks case and define an "input energy" for the system. Passivity is then defined in terms of a nonnegativity condition on this input energy.

One of the main motivations for studying passivity in the control theory context has been its connection with stability [3],[4]. An especially important result in this area is the Kalman-Yakubovich-Popov lemma [5]-[7] or Positive Real Lemma, which was used in solving the well-known stability problem of Lur'e [8]. In essence, this lemma states that a transfer function is positive real if and only if a certain set of matrix equations has a nonnegative definite solution. The connection between positive real functions and passive systems is, of course, well known [9]. Generalizations of the positive real lemma are now available which extend the results to positive real matrices [5],[10], and in particular Anderson [11] has obtained a solution to the multivariable version of the problem of Lur'e. Applications of the positive real lemma also include spectral factorization [12], network synthesis [21], and the solution of an inverse problem of linear optimal control [13].

The main limitation of the positive real concept is that it applies only to linear systems. For nonlinear systems, the most promising approach to studying the properties of passive systems appears to be that of Willems [14]. The starting point in Willems' theory is the definition of a "dissipative" system in terms of an inequality involving, in effect, the stored energy of the system. Passivity, as defined in this paper, then appears as a special case of dissipativeness.

The results of this paper lie somewhere between those for linear systems and those for the extremely general class of systems studied by Willems. The principal restriction imposed is that the state equations for the systems considered should involve the control vector only linearly. (This condition is less restrictive than it might at first appear to be. Balakrishnan [22] has shown that, under very mild restrictions, there exists a choice of state vector for almost any controllable finite-dimensional system such that the control appears linearly in the state equations.) By making this restriction, it turns out to be possible to obtain results that are considerably more explicit than have hitherto been available for nonlinear systems. In fact, the central result could be interpreted as a nonlinear version of the positive real lemma.

This result, given as Theorem 1 in Section II, is the cornerstone of the subsequent development; as with the positive real lemma, it shows that a system is passive if and only if there exists a certain scalar function of the state which is nonnegative. The remainder of Section II is devoted to showing that this function has the properties of stored energy, thus arriving by a different route at some of the results of [14].

In Sections III–V the results of Section II are applied to a class of inverse optimal control problems, a class of singular control problems, and a nonlinear extension of the Popov stability problem [8],[15]. All of these results are believed to be new, although a restricted version of the results of Section III has been reported in [16]. The algorithm of Section IV is particularly interesting, in that it achieves a partial solution of a singular optimal control problem by methods which are equally applicable to nonsingular problems. It appears that these ideas could be extended considerably, although the details still remain to be worked out.

II. An Algebraic Condition for Passivity

The systems to be studied in this paper are described by the equations

$$\dot{x} = f(x) + G(x)u$$
$$y = h(x) + J(x)u \tag{1}$$

where $f(\cdot)$ and $h(\cdot)$ are real vector functions of the state vector x, with $f(0) = 0$, $h(0) = 0$, and $G(\cdot)$ and $J(\cdot)$ are real matrix functions of x. In order to guarantee the existence of solutions to the equations to be introduced below, it will be assumed that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$ all possess continuous derivatives of all orders, although weaker conditions would probably suffice. The input vector u and the output vector y have the same dimension, so that $J(x)$ is a square matrix. The system is assumed to be completely controllable, in the sense that for any finite states x_0 and x_1, there exists a finite time t_1 and a square-integrable control $u(t)$ defined on $[0,t_1]$ such that the state can be driven from $x(0) = x_0$ to $x(t_1) = x_1$. In addition, a form of local controllability is assumed; for any x_0 and any x_1 in a suitably small open neighborhood of x_0, there exists a choice of $u(\cdot)$ and t_1 as above with the additional property that

$$\left|\int_0^{t_1} u'(t)y(t)\,dt\right| \le \rho(\|x_1 - x_0\|)$$

for some continuous $\rho(\cdot)$ such that $\rho(0) = 0$.

The dynamical system (1) will be called *passive* if, whenever $x(t_0) = 0$ (for any t_0),

$$\int_{t_0}^{T} 2u'(t)y(t)\,dt \ge 0 \tag{2}$$

for all $T \ge t_0$ and all square-integrable $u(t)$. (The multiplier 2 is simply a scaling factor which it will be convenient to use in the following equations.) The quantity on the left-hand side of (2) will sometimes be called the *input energy*.

An alternative formulation of condition (2) is given in the following theorem.

Theorem 1: A necessary and sufficient condition for system (1) to be passive is that there exist real functions $\phi(\cdot)$, $l(\cdot)$, and $W(\cdot)$, with $\phi(x)$ continuous and with

$$\phi(x) \ge 0, \qquad \text{for all } x$$

and

$$\phi(0) = 0$$

such that

$$\nabla'\phi(x)f(x) = -l'(x)l(x)$$
$$\tfrac{1}{2}G'(x)\nabla\phi(x) = h(x) - W'(x)l(x)$$
$$J(x) + J'(x) = W'(x)W(x). \tag{3}$$

Moreover, if J is a constant matrix, then W may be taken to be constant.

Proof:

1) *Sufficiency:* Suppose $\phi(\cdot)$, $l(\cdot)$, and $W(\cdot)$ are given such that (3) are satisfied. Then for any square-integrable $u(t)$, and any t_0 and $T \ge t_0$, and any $x(t_0)$,

$$\int_{t_0}^{T} 2u'(t)y(t)\,dt = \int_{t_0}^{T} \{2u'h(x) + u'[J(x) + J'(x)]u\}\,dt$$
$$= \int_{t_0}^{T} [u'G'(x)\nabla\phi(x) + 2u'W'(x)l(x) + u'W'(x)W(x)u]\,dt$$
$$= \int_{t_0}^{T} \{\nabla'\phi(x)[f(x) + G(x)u] + 2u'W'(x)l(x) + u'W'(x)W(x)u - \nabla'\phi(x)f(x)\}\,dt$$
$$= \int_{t_0}^{T} \left\{\frac{d}{dt}\phi[x(t)] + 2u'W'(x)l(x) + u'W'(x)W(x)u + l'(x)l(x)\right\} dt$$
$$= \phi[x(T)] - \phi[x(t_0)] + \int_{t_0}^{T} [l(x) + W(x)u]'[l(x) + W(x)u]\,dt.$$

Setting $x(t_0) = 0$, the result follows.

2) *Necessity:* As before, assume that the system (1) is started in the initial state $x(t_0) = 0$. From the definition of $y(t)$, it follows that

$$\int_{t_0}^{T} 2u'(t)y(t)\,dt = \int_{t_0}^{T} \{2u'h(x) + u'[J(x) + J'(x)]u\}\,dt.$$

If the system is passive, this integral should be nonnegative for any $T \geq t_0$ and any $u(\cdot)$. This implies that $[J(x) + J'(x)]$ is a nonnegative definite matrix for all x, since otherwise one could find[1] a $u(t)$ such that the integral became negative for some T. Consequently, the symmetric matrix $[J(x) + J'(x)]$ admits of a (nonunique) factorization

$$J(x) + J'(x) = W'(x)W(x). \tag{4}$$

Now for any state x_0, there exists by controllability a time $t_0 < 0$ and a control $u(\cdot)$ defined on the interval $[t_0,0]$ such that $x(0) = x_0$. By passivity

$$\int_{t_0}^{0} 2u'(t)y(t)\,dt + \int_{0}^{T} 2u'(t)y(t)\,dt \geq 0.$$

That is

$$\int_{0}^{T} 2u'(t)y(t)\,dt \geq -\int_{t_0}^{0} 2u'(t)y(t)\,dt.$$

The right-hand side of this inequality depends only on x_0, whereas $u(t)$ can be chosen arbitrarily for $t \geq 0$. There therefore exists a function $C(x_0)$ of x_0 alone such that

$$\int_{0}^{T} 2u'(t)y(t)\,dt \geq -C(x_0) \tag{5}$$

whenever $x(0) = x_0$.

Now define $\phi(x)$ as

$$\phi(x_0) = -\lim_{T\to\infty} \inf_{u(\cdot)} \int_{0}^{T} 2u'(t)y(t)\,dt \tag{6}$$

subject to (1) and the boundary condition $x(0) = x_0$. Existence of $\phi(x)$ follows[2] from the inequality (5); moreover, it is clear that

$$\phi(0) = 0$$

and

$$\phi(x) \geq 0, \qquad \text{for all } x.$$

Let us suppose temporarily that $[J(x) + J'(x)]$ is nonsingular for all x. Then, performing the optimization indicated in (6) above, it is easily shown[3] that $\phi(x)$ satisfies the (Hamilton–Jacobi) equation

$$\nabla'\phi(x)f(x) + [h(x) - \tfrac{1}{2}G'(x)\nabla\phi(x)]'(J + J')^{-1}[h(x) - \tfrac{1}{2}G'(x)\nabla\phi(x)] = 0.$$

If $W(x)$ in (4) is chosen to be the positive definite square root of $(J + J')$, then a function $l(x)$ may be defined as

$$l(x) = (W')^{-1}[h(x) - \tfrac{1}{2}G'(x)\nabla\phi(x)]$$

from which (3) follows.

If $(J + J')$ is singular, the above procedure must be slightly modified. In this case, a new matrix $J_\epsilon(x)$ is defined as

$$J_\epsilon(x) = J(x) + \tfrac{1}{2}\epsilon I$$

for some small positive constant ϵ. Functions $\phi_\epsilon(x)$, $l_\epsilon(x)$, and $W_\epsilon(x)$ can then be defined in the obvious manner. From (4), (5), and (6), it is then straightforward to establish the existence of the limits

$$W(x) = \lim_{\epsilon\to 0} W_\epsilon(x)$$
$$\phi(x) = \lim_{\epsilon\to 0} \phi_\epsilon(x)$$

and

$$l(x) = \lim_{\epsilon\to 0} l_\epsilon(x).$$

Since these limits are approached continuously, it is simple to establish that (3) are in fact satisfied. ∇∇∇

The significance of Theorem 1 is that conditions for passivity—essentially a property of the input-output relationship of a system— have been stated in terms of functions of the state vector; in a sense, (3) are conditions on the internal structure of the system. It is worth noting

[1] The construction is roughly as follows. First, find a control driving x to a point where $(J + J')$ is no longer nonnegative definite; then follow this with a "pulse" in u large enough to overcome any positive contribution to the integral.

[2,3] The details are not entirely trivial. However, the argument follows the same lines as, say, [16].

that (3) would remain unchanged if $\phi(x)$ where changed by any constant amount. That is, the constraint $\phi(0) = 0$ could have been dropped, and the condition $\phi(x) \geq 0$ replaced by $\phi(x) \geq \phi(0)$. To avoid trivia, however, only those solutions for which $\phi(0) = 0$ will be considered in the sequel.

If the physical significance of the vectors $u(t)$ and $y(t)$ are such that the integral in (2) represents input energy, one can go even further. Recall that

$$\int_{t_0}^{T} 2u'(t)y(t)\,dt = \phi[x(T)] - \phi[x(t_0)] + \int_{t_0}^{T} [l(x) + W(x)u]'[l(x) + W(x)u]\,dt \quad (7)$$

which may be interpreted as the "conservation of energy" equation

Input energy = Final energy − Initial energy + Dissipated energy.

The nonnegative quantity $\phi(x)$ then appears as the *stored energy* of the system, while the second integral corresponds to dissipated energy. As expected for passive systems, the dissipated energy is always nonnegative, although it is path-dependent. The change in stored energy while moving from state $x(t_0)$ to $x(T)$ is, of course, independent of the path taken.

Since no physical significance has yet been attached to the state vector x, the stored energy is not, in general, uniquely specified by the state equations (1). Typically, (3) will have a large number of solutions. However, it is possible to place bounds on these solutions, as shown in the following theorem.

Theorem 2: Equations (3) possess a maximum and a minimum solution, in the sense that there exist solutions $\phi_a(x)$ and $\phi_r(x)$ such that

$$\phi_a(x) \leq \phi(x) \leq \phi_r(x), \qquad \text{for all } x$$

where $\phi(x)$ is any solution of (3). Moreover,

$$\phi_a(x_0) = -\lim_{T\to\infty} \inf_{u(\cdot)} \int_0^T 2u'(t)y(t)\,dt$$

subject to (1) and $x(0) = x_0$; and

$$\phi_r(x_0) = \lim_{t_0\to-\infty} \inf_{u(\cdot)} \int_{t_0}^{0} 2u'(t)y(t)\,dt$$

subject to (1), $x(t_0) = 0$, and $x(0) = x_0$.

Proof: From (7), every solution set $\{\phi(\cdot),l(\cdot),W(\cdot)\}$ of (3) satisfies

$$\int_{t_0}^{T} 2u'(t)y(t)\,dt = \phi[x(T)] - \phi[x(t_0)] + \int_{t_0}^{T} [l(x) + W(x)u]'[l(x) + W(x)u]\,dt \quad (7)$$

where $x(t)$ and $y(t)$ are solutions of (1). The left side of this equation depends on $u(\cdot)$ and the boundary conditions on $x(t)$, but is independent of the particular solution set $\{\phi(\cdot),l(\cdot),W(\cdot)\}$.

Let $\{\phi_i(\cdot),l_i(\cdot),W_i(\cdot)\}$, $i = 1,2$, be two sets of solutions of (3), and for simplicity define "dissipation outputs"

$$y_i(t) = l_i[x(t)] + W_i[x(t)]u(t).$$

Then (7) implies that

$$\phi_1[x(T)] - \phi_1[x(t_0)] + \int_{t_0}^{T} y_1'(t)y_1(t)\,dt = \phi_2[x(T)] - \phi_2[x(t_0)] + \int_{t_0}^{T} y_2'(t)y_2(t)\,dt \quad (8)$$

for any $u(\cdot)$ and any boundary conditions $x(t_0),x(T)$.

Let $\phi_1(x)$ be $\phi_a(x)$. Then from the definition of $\phi_a(x)$, there exists a sequence of controls $\{u^{(j)}(\cdot)\}$ such that, with $x(0) = x_0$,

$$\lim_{T\to\infty} \lim_{j\to\infty} \int_0^T 2u^{(j)\prime}(t)y^{(j)}(t)\,dt = -\phi_1(x_0)$$

where $y^{(j)}(t)$ is the corresponding solution of (1). Thus,

$$\lim_{T\to\infty} \lim_{j\to\infty} \left\{\phi_1[x^{(j)}(T)] + \int_0^T y_1^{(j)\prime}(t)y_1^{(j)}(t)\,dt\right\} = 0$$

where $x^{(j)}(t)$ and $y_1^{(j)}(t)$ are defined similarly.

Equation (8) can now be rearranged to give

$$\phi_2(x_0) = \phi_1(x_0) + \lim_{T\to\infty} \lim_{j\to\infty} \left\{\phi_2[x^{(j)}(T)] + \int_0^T y_2^{(j)\prime}(t) \cdot y_2^{(j)}(t)\,dt\right\} - \lim_{T\to\infty} \lim_{j\to\infty} \left\{\phi_1[x^{(j)}(T)] + \int_0^T y_1^{(j)\prime}(t)y_1^{(j)}(t)\,dt\right\}.$$

The second limit is zero, from above, whereas the first is nonnegative. Therefore,

$$\phi_2(x_0) \geq \phi_1(x_0), \qquad \text{for all } x_0.$$

That is,

$$\phi(x) \geq \phi_a(x), \qquad \text{for all } x$$

where $\phi(\cdot)$ is any solution of (3). This completes the proof that $\phi_a(x)$ is the minimum solution. That $\phi_r(x)$ is the maximum solution may be shown in a similar manner. ∇∇∇

In the terminology of Willems [4], $\phi_a(x)$ is the *available energy* of the system (1)—by definition, it is the maximum amount of energy that can be extracted at the terminals when starting from the initial state x. Similarly, $\phi_r(x)$ may be called the *required energy*, since it is the minimum energy required to excite the system to a state x from the equilibrium (zero-energy) state. These two functions will be identical if the system is reversible, [4].

Theorem 2 shows that all solutions $\phi(x)$ of (3) lie in a bounded set. It is also worth noting that this set is convex.

Lemma 1: If $\phi_1(x)$ and $\phi_2(x)$ are any two solutions of (3), then

$$\hat{\phi}(x) = \alpha\phi_1(x) + (1 - \alpha)\phi_2(x)$$

(for any α such that $0 \leq \alpha \leq 1$) is also a solution.

Proof: Corresponding to the solutions $\phi_1(x)$ and $\phi_2(x)$, there are vectors $l_1(x)$ and $l_2(x)$, and matrices $W_1(x)$ and $W_2(x)$, appearing in (3). Now define,

$$\hat{l}(x) = \begin{bmatrix} \sqrt{\alpha} l_1(x) \\ \sqrt{1-\alpha} l_2(x) \end{bmatrix}; \quad \hat{W}(x) = \begin{bmatrix} \sqrt{\alpha} W_1(x) \\ \sqrt{1-\alpha} W_2(x) \end{bmatrix}.$$

Then $\hat{\phi}(x)$ is readily seen to be a solution of (3), with $l(x) = \hat{l}(x)$ and $W(x) = \hat{W}(x)$. ∇∇∇

Actually, the convexity of the set of energy functions has already been proved by Willems [14]. The novel point of Lemma 1 is that it is possible to explicitly exhibit the associated dissipation terms $\hat{l}(x)$ and $\hat{W}(x)$.

In a later section of this paper, the functions $\phi(x)$ will be used as Lyapunov functions; it is therefore of interest to note conditions under which $\phi(x)$ will be positive definite (in the sense that $\phi(x) = 0$ implies $x = 0$).

Definition: The system (1) will be called *observable* if, for any trajectory such that $u(t) \equiv 0$, $y(t) \equiv 0$ implies $x(t) \equiv 0$.

(Note that this is a weaker condition than is required by more standard definitions of observability. Observability in the above sense does not imply the ability to deduce the initial state from output measurements alone, although it does imply the ability to detect the presence of a nonzero state.)

Lemma 2: If the system (1) is passive and observable, then all solutions $\phi(x)$ of (3) are positive definite.

Proof: Suppose that there exists some x_0 such that $\phi(x_0) = 0$, and let $x(t)$ be the solution of

$$\dot{x}(t) = f[x(t)], \qquad x(0) = x_0.$$

Then, from (3),

$$\frac{d}{dt}\phi[x(t)] = \nabla'\phi(x)f(x) = -l'(x)l(x). \tag{9}$$

Therefore,

$$\phi[x(t)] \leq \phi[x_0], \qquad \text{for } t \geq 0.$$

It follows that $\phi[x(t)]$ is identically zero along this trajectory, so that $l[x(t)]$ is also zero. Since $\phi(x)$ is a continuous nonnegative function of x, it also follows that $\nabla\phi(x)$ is zero along the same trajectory. From the second of equations (3), then,

$$h[x(t)] \equiv 0$$

along the same trajectory.

Unless $x_0 = 0$, this contradicts the assumed observability of (1). It must therefore be true that $\phi(x)$ is positive definite. ∇∇∇

Corollary: If the system (1) is passive and observable, then the free system

$$\dot{x} = f(x)$$

is (Lyapunov) stable.

Proof: This follows directly from (9). ∇∇∇

Note, however, that *asymptotic* stability has not been proved. This generally requires stronger conditions than have been noted above.

The preceding discussion has shown that there is a natural ordering of the solutions of (3), corresponding to an ordering in the stored energy $\phi(x)$. As an alternative, one could define an ordering of solutions in terms of dissipated energy. This approach is illustrated below.

Definition: With $\{\phi_i(x), l_i(x), W_i\}$ any solution set of (3), let

$$y_i = \Gamma_i\{u\}$$

represent the input-to-dissipation output relationship of the system

$$\begin{aligned} \dot{x} &= f(x) + G(x)u \\ y_i &= l_i(x) + W_i(x)u. \end{aligned} \tag{10}$$

Then Γ_1 has less delay than Γ_2, written $\Gamma_1 < \Gamma_2$, if, for any time $t_1 \geq t_0$, and any common input $u(\cdot) \epsilon L_2[t_0,t_1]$,

$$\int_{t_0}^{t_1} y_1'(t)y_1(t)\,dt \geq \int_{t_0}^{t_1} y_2'(t)y_2(t)\,dt \tag{11}$$

whenever $x(t_0) = 0$, and if equality does not hold for all $u(t)$ and all t_1. The notation $\Gamma_1 \leq \Gamma_2$ will be used in case (11) holds, with the possibility that equality always holds.

The relation $\leq$ represents a partial ordering, based on the amount of energy dissipated along any given trajectory.

Theorem 3: Let Γ_1 and Γ_2 be any two members of the family (10). Then $\Gamma_1 \leq \Gamma_2$ if and only if $\phi_1(x) \leq \phi_2(x)$ for all x. Moreover, regardless of any ordering between Γ_1 and Γ_2,

$$\int_{t_0}^{T} y_1'(t)y_1(t)\,dt = \int_{t_0}^{T} y_2'(t)y_2(t)\,dt \tag{12}$$

for any $u(\cdot)$ such that $x(T) = x(t_0)$.

Proof: The proof is obvious from (8). ∇∇∇

The meaning of the description "less delay" should now be clear. For any common input to the two systems, the dissipation responses have the same mean square value (see (12)) but (11) shows that the main part of the response occurs earlier for the first system than for the second. In particular, the system whose stored energy function is $\phi_a(x)$ is a *minimum delay* system. At the other extreme, the system whose stored energy is $\phi_r(x)$ is a *maximum delay* system.

For linear systems, the concept of "minimum delay" actually turns out to correspond with the classical notion of "minimum phase." This special case has been previously studied by Anderson [17], in the context of spectral factorization of power spectrum matrices. Reference [17] also establishes that there is a sense in which the term "maximum phase" could be applied to maximum delay systems, as defined here.

III. Application to Optimal Control

A typical problem in optimal control theory is the following:

Given the system equations

$$\dot{x} = f(x) + Gu \tag{13}$$

where $f(\cdot)$ is a real function of the vector x possessing continuous derivatives of all orders, $f(0) = 0$, and G is a constant matrix, find a control $u(t)$ that will minimize the performance index

$$V_1 = \lim_{T\to\infty} \left\{ n[x(T)] + \int_0^T [l'(x)l(x) + u'u]\, dt \right\} \tag{14}$$

where $n(x) \geq 0$ for all x. Without substantial loss of generality, it may be assumed that $l(0) = 0$ and $n(0) = 0$.

The optimal control will in general be of the form

$$u^* = -k(x) \tag{15}$$

and the minimum value of the performance index will be some function $\phi(x_0)$ of the initial state x_0. However, it may well be that there exist many different functions $l(x)$, and correspondingly different $\phi(x)$, for which the same control law (15) is optimal. If the system (13) and control law (15) are given, we shall say that the pair $\{\phi(\cdot),l(\cdot)\}$ is a solution to the *inverse problem* if the performance index (14)—for some nonnegative choice of $n(x)$—is minimized by (15), the minimum value being $\phi(x_0)$.

Necessary and sufficient conditions for the existence of a solution to the inverse problem have been given in [16]. For linear systems with linear state feedback, one can in fact obtain extremely detailed results, the best-known results being those of Kalman [18]. However, these references are primarily concerned with obtaining one solution to the inverse problem. A method of generating *all* solutions is indicated in the following lemma.

Lemma 3: Suppose that the system equations (13) and control law (15) are given. Then a pair $\{\phi(\cdot),l(\cdot)\}$ is a solution to the inverse problem if and only if $\phi(x)$ and $l(x)$ satisfy the equations

$$\nabla'\phi(x)[f(x) - \tfrac{1}{2}Gk(x)] = -l'(x)l(x)$$

$$\tfrac{1}{2}G'\nabla\phi(x) = k(x)$$

$$\phi(0) = 0$$

and

$$\phi(x) \geq 0, \qquad \text{for all } x.$$

Proof: 1) Suppose $\{\phi(\cdot),l(\cdot)\}$ is known to be a solution of the inverse problem. Then $k(x)$ is given [16] from standard Hamilton–Jacobi theory as

$$k(x) = \tfrac{1}{2}G'\nabla\phi(x)$$

which shows that the second equation is satisfied. Also, [16], $\phi(x)$ must be a nonnegative solution of the Hamilton–Jacobi equation (in its limiting form)

$$\nabla'\phi(x)f(x) - \tfrac{1}{4}\nabla'\phi(x)GG'\nabla\phi(x) + l'(x)l(x) = 0$$

from which the first equation follows.

2) Suppose that $\phi(x)$ and $l(x)$ are known to satisfy the above equations. Then it is easily shown that, for any square-integrable $u(\cdot)$,

$$\int_0^T [l'(x)l(x) + u'u]\, dt = \phi[x(0)] - \phi[x(T)] + \int_0^T [u + k(x)]'[u + k(x)]\, dt.$$

Setting $n(x) = \phi(x)$, the result follows. ∇∇∇

In view of Theorems 1 and 2, these results may be summarized as follows.

Theorem 4: A necessary and sufficient condition for the existence of a solution to the inverse problem is that the system

$$\dot{x} = f(x) - \tfrac{1}{2}Gk(x) + Gu$$

$$y = k(x)$$

be passive. If this condition is satisfied, then there exist two solutions $\{\phi_a(\cdot),l_a(\cdot)\}$, and $\{\phi_r(\cdot),l_r(\cdot)\}$, not necessarily distinct, such that all other solutions $\{\phi(\cdot),l(\cdot)\}$ satisfy the constraint

$$\phi_a(x) \leq \phi(x) \leq \phi_r(x), \qquad \text{for all } x.$$

IV. Singular Optimization Problems

As a further application of the results of Section II, consider the following problem:
with $x(t)$ governed by

$$\dot{x} = f(x) + G(x)u, \qquad x(0) = x_0 \tag{16}$$

find $u(\cdot)$ such that the performance index

$$V_2 = \lim_{T\to\infty} \int_0^T [l(x) + W(x)u]'[l(x) + W(x)u]\, dt \tag{17}$$

is minimized. In addition to the state equation (16), there may be boundary conditions on x and possibly some constraints on u.

If $W'(x)W(x)$ is invertible for all x, the solution of the above problem is relatively straightforward. If, on the other hand, $W'(x)W(x)$ is singular, most standard optimization methods fail to find an optimal solution. Although it is straightforward to establish that the optimal control is bang-bang (that is, the control attains its limits, or is infinite if there are no *a priori* bounds) for most values of x, there is the possibility of *singular regions* where the optimal control is determined by criteria independent of the bounds on u. The determination of the location of the singular regions in the state space is usually the most difficult step in solving the above problem.

For the purposes of this section, it will be assumed that there are no *a priori* bounds on u, and that boundary conditions for x have not yet been specified. That is, we wish to derive the optimal controls for all possible combinations of boundary conditions. A *singular region* will then be defined as any region in (x,u) space in which the optimal control is finite. (This differs from the usual definition, in that the case where $W'(x)W(x)$ is invertible has not been excluded.) Depending on the actual constraints on u and on the boundary conditions, some of these regions may not in fact enter into the optimal solution, but these considerations are most easily handled after the singular regions have been located.

An obvious candidate as a singular region is the region

$$l(x) + W(x)u = 0.$$

However, trajectories in this region may not have the property that the boundary conditions are met. Moreover, the region may be a trivial one, consisting only of the origin. To proceed further, it is obviously necessary to find *all* the singular regions, and then to exclude those which do not meet the side constraints.

In the sequel, it will be assumed that the free system

$$\dot{x} = f(x), \qquad x(0) = x_0 \tag{18}$$

is asymptotically stable. [Unstable systems can also be handled if it can be shown that there exists a stabilizing feedback control law[4] for the original system (16). Supposing that this control is of the form $-k(x)$, a new control can be defined via

$$u_1 = u - k(x)$$

and (16) and (17) modified appropriately. This does not change the form of (16) or (17).] With this assumption, define[5] a function $\phi(x)$ via

$$\phi(x_0) = \int_0^\infty l'(x)l(x)\, dt$$

[where $x(t)$ is governed by (18)], and then define functions $h(x)$ and $J(x)$ as

$$h(x) = \tfrac{1}{2}G'(x)\nabla\phi(x) + W'(x)l(x)$$
$$J(x) = \tfrac{1}{2}W'(x)W(x).$$

The significance of these functions is shown in the following lemma.

Lemma 4: The system

$$\dot{x} = f(x) + G(x)u$$
$$y = h(x) + J(x)u$$

is passive.

Proof: From the definition of $\phi(x)$, it follows that

$$\nabla'\phi(x)f(x) = -l'(x)l(x)$$

and that

$$\phi(x) \geq 0, \qquad \text{for all } x.$$

Equations (3) are therefore trivially satisfied, and Theorem 1 may be applied. ∇∇∇

A number of singular regions may now be derived as follows.

Theorem 5: With $h(x)$ and $J(x)$ defined as above, let $\{\phi_i(x), l_i(x), W_i(x)\}$ be *any* solution set of the equations

$$\nabla'\phi_i(x)f(x) = -l_i'(x)l_i(x)$$
$$\tfrac{1}{2}G'(x)\nabla\phi_i(x) = h(x) - W_i'(x)l_i(x)$$
$$W_i'(x)W_i(x) = J(x) + J'(x). \tag{19}$$

Then the region

$$l_i(x) + W_i(x)u = 0 \tag{20}$$

is a singular region.

Proof: By defining a new function

$$\hat{\phi}(x) = \phi(x) - \phi_i(x)$$

the following set of equations may be derived:

$$\nabla'\hat{\phi}(x)f(x) = l_i'(x)l_i(x) - l'(x)l(x)$$
$$\tfrac{1}{2}G'(x)\nabla\hat{\phi}(x) = W_i'(x)l_i(x) - W'(x)l(x)$$
$$W_i'(x)W_i(x) = W'(x)W(x).$$

Then, with $x(t)$ given by (16),

$$\frac{d}{dt}\hat{\phi}[x(t)] = \nabla'\hat{\phi}(x)f(x) + \nabla'\hat{\phi}(x)G(x)u$$

for any $u(t)$. With the aid of the above equations, this reduces to

$$\frac{d}{dt}\hat{\phi}[x(t)] = [l_i(x) + W_i(x)u]'[l_i(x) + W_i(x)u] - [l(x) + W(x)u]'[l(x) + W(x)u].$$

Therefore, for any times t_0 and $t_1 \geq t_0$,

$$\int_{t_0}^{t_1} [l(x) + W(x)u]'[l(x) + W(x)u]\, dt = \hat{\phi}[x(t_0)] - \hat{\phi}[x(t_1)] + \int_{t_0}^{t_1} [l_i(x) + W_i(x)u]'[l_i(x) + W_i(x)u]\, dt.$$

Equation (20) is therefore a sufficient condition for optimality, provided that the boundary conditions are such that $\hat{\phi}[x(t_1)]$ is also minimized. It follows that (20) describes a singular region. ∇∇∇

Note, incidentally, that the above theorem holds whether or not $W'(x)W(x)$ is invertible. If this matrix is in fact invertible, then the "singular region" covers the entire state space, and the solution for u reduces to that found for nonsingular problems. In all other cases, (20) can be satisfied only in a subset of the state space. It is felt that a major contribution of this section is that Theorem 5 covers singular and nonsingular problems equally well, by considering nonsingular problems in x space as "singular" problems in (x,u) space.

V. Tolerance of Sector Nonlinearities

In this section we investigate the stability of the system

$$\dot{x}(t) = f[x(t)] + G\psi\{k[x(t)]\} \tag{21}$$

where $f(\cdot)$ and $k(\cdot)$ are known linear or nonlinear functions, G is a known matrix, and $\psi(\cdot)$ is an unknown nonlinearity. However, $\psi(\cdot)$ is partly constrained by the conditions

[4] Controllability appears to be a sufficient condition for such a control law to exist [16].

[5] Boundedness of $\phi(\cdot)$ needs to be explicitly assumed.

$$\sigma'[\psi(\sigma) - T\sigma] \geq [\psi(\sigma) - T\sigma]'M[\psi(\sigma) - T\sigma] + \epsilon\sigma'\sigma, \quad \text{for all } \sigma \neq 0 \quad (22)$$

and

$$\psi(0) = 0.$$

In the above inequality, T and M are known constant matrices, with M nonnegative and symmetric, and ϵ is a small positive scalar constant.

In the case of independent nonlinearities, i.e., the jth component of $\psi(\sigma)$ depends only on the jth component of σ, we may take T and M to be diagonal. The inequality then reduces to a set of inequalities of the form

$$t_{ii} < \frac{\psi_i(\sigma)}{\sigma_i} < t_{ii} + \frac{1}{m_{ii}}.$$

That is, the graph of $\psi_i(\sigma)$ as a function of σ_i is known to lie in the sector bounded by straight lines of slope t_{ii} and $t_{ii} + (1/m_{ii})$. In general, then, we call $\psi(\sigma)$ a *sector nonlinearity*.

The main result is as follows.

Theorem 6: If the system

$$\dot{x} = f(x) - GTk(x) + Gu$$
$$y = k(x) + Mu$$

is passive and observable, then the system (21) is asymptotically stable.

Proof: The conditions of the theorem imply the existence of $\{\phi(\cdot), l(\cdot), W\}$, with $\phi(\cdot)$ positive definite, such that

$$\nabla'\phi(x)f(x) - \nabla'\phi(x)GTk(x) = -l'(x)l(x)$$
$$\tfrac{1}{2}G'\nabla\phi(x) = k(x) - W'l(x)$$
$$M + M' = W'W.$$

With $x(t)$ governed by (21), it follows easily that

$$\begin{aligned}\frac{d\phi[x(t)]}{dt} &= \nabla'\phi(x)f(x) - \nabla'\phi(x)G\psi[k(x)] \\ &= -\{l(x) - W\psi[k(x)] + WTk(x)\}'\{l(x) \\ &\quad - W\psi[k(x)] + WTk(x)\} \\ &\quad - 2\{k'(x)[\psi[k(x)] - Tk(x)] \\ &\quad - [\psi[k(x)] - Tk(x)]'M[\psi[k(x)] - Tk(x)]\}.\end{aligned}$$

The derivative is clearly nonnegative, so that $\phi(x)$ is a Lyapunov function establishing stability for (21). Moreover, the above derivative can never be identically zero along a nontrivial trajectory, for by the inequality (22) this would imply that $k(x) \equiv 0$ along the same trajectory; this possibility is ruled out by observability. It then follows [19] that

$$\lim_{t\to\infty} \phi[x(t)] = 0$$

which is sufficient to prove asymptotic stability. ∇∇∇

VI. Conclusions

The central point of this paper is of course, Theorem 1. Here, it is shown that passivity of a certain class of systems is equivalent to the existence of a state function $\phi(x)$ satisfying a set of well-defined equations. The significance of this theorem is that the *input-output* property of passivity may be replaced by a set of constraints on the *internal* structure of a system. The interpretation of $\phi(x)$ as an energy function is in a sense a secondary issue, but it allows some useful insights into the meaning of Theorem 1.

As with the more general storage functions of Willems [4], [14], it turns out that the stored energy is nonunique; this implies, among other things, that when one is interested in computing internal energy the state equations provide an incomplete description of a system. Consider for example the case of linear systems. It is natural to call two systems *equivalent* [20] if their state vectors $x(t)$ and $\hat{x}(t)$ are such that $\hat{x}(t) = Tx(t)$ for all t, for some invertible matrix T. Supposing that the stored energies for the two systems are given by $x'Px$ and $\hat{x}'\hat{P}\hat{x}$, then it is *not* necessarily true that $P = T'\hat{P}T$. This conclusion has a natural and fairly obvious physical interpretation in the case of electrical networks, but the implications for more general control systems are not entirely clear. It would be of considerable interest to develop a theory of control systems which took account of the differences between different systems having the same state equations.

The applications listed in Sections III–V are intended only as a brief survey; it is clear that one could obtain stronger or more general results for the problems mentioned in these sections. For example, the Lyapunov function of Section V could be augmented by an integral term involving the unknown nonlinearity (as was done in [11]), with a corresponding increase in generality of the system of Theorem 6. Another area where the theory shows promise is in the synthesis of nonlinear electrical networks; the concepts of stored and dissipated energy are readily translated into physical terms, and the only real problem is to determine how the network elements should be interconnected. Yet another application is in problems involving the stability of interconnected systems—the work of Zames [3] is especially relevant here. The precise details in these last two cases still remain to be worked out.

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Autonomous Periodic Motion in Nonlinear Feedback Systems

ERIK J. L. NOLDUS

***Abstract*—Sufficient conditions are presented for the existence of periodic motions in a class of autonomous, time-invariant, nonlinear feedback systems. The conditions can be interpreted as circle type criteria, stated in terms of the frequency response and root locus diagrams for the system's linear part, and in terms of the characteristics of the nonlinearity.**

I. Introduction

THE present paper is concerned with establishing the existence of nontrivial periodic solutions, for the differential equation of an autonomous, time-invariant, nonlinear feedback system. While a general theory of nonlinear oscillations is available for second-order systems, where analytical-topological methods may be applied very nicely, the problem remains a challenging one for systems of order higher than two. For its solution, only a few basic principles can be found in the textbooks. Truly, some results of this kind, for specific cases of third-order systems, have been known for a relatively long time, one much cited example by Rauch [1] going back as far as 1950. Other examples can be found in the Russian literature [2]. It seems difficult however to derive a simple criterion that applies to the case of a general system of order n, containing an arbitrary nonlinear amplifier.

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