Invert again the resulting matrix to remove two serial capacitances and after a last inversion apply a new decomposition of the admittance matrix to get finally a bridged-T network.

For the matrix b a single inversion gives a T-network. The resulting network is given on Fig. 1. Note that the number of elements of this synthesis is 13, i.e., 28-percent less than in the synthesis proposed by Lucal.

If we had chosen an alternate optimal solution $X_{\text{opt}} = (31/36, 1, 1)$ we whould have obtained a synthesis with 15 elements, i.e., 17-percent less than in Lucal's synthesis.

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Preliminary Simplifications in State-Space Impedance Synthesis

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Abstract—**State-space techniques for impedance matrix synthesis can be difficult to apply when the prescribed impedance matrix $Z(s)$ is such that $Z(\infty) + Z'(\infty)$ is singular. Here this problem is overcome by a sequence of preliminary lossless extractions. In the case of lossless networks, the procedure reduces to a multiport Cauer synthesis.**

I. Introduction

IN RECENT YEARS, it has become clear that synthesis of linear passive networks, in particular multiport networks, may well be handled more conveniently by state-space methods [1] than by the better known classical frequency-domain methods [2]. The basic difference between the two classes of methods is that the classical approach seeks a direct synthesis of a prescribed immittance matrix $Z(s)$, while the state-space approach seeks instead a synthesis of the dynamical system

$$\dot{x} = Fx + Gu$$
$$y = H'x + Ju. \tag{1}$$

The matrices of (1) are related to $Z(s)$ by

$$Z(s) = J + H'(sI - F)^{-1}G. \tag{2}$$

Manuscript received June 11, 1973; revised July 12, 1973. This work was supported by the Australian Research Grants Committee.
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Moreover, the computation of these matrices from $Z(s)$ is a relatively straightforward procedure (see, for example, [1]).

Virtually all of the state-space synthesis methods rely ultimately on the following lemma [3], [4].

Lemma: Suppose that (1) is a minimal [5] realization of $Z(s)$. Then the system (1) is passive (equivalently, $Z(s)$ is positive-real), if and only if there exist real matrices P, L, and W, with P positive definite, such that

$$PF + F'P = -LL'$$
$$PG = H - LW$$
$$W'W = J + J'. \tag{3}$$

The key element in synthesis is the solution of (3). (A number of solution methods are surveyed in [1].) However, the equations can be difficult to solve if $(J + J')$ is singular, which tends to negate the basic simplicity of the state-space approach. The equivalent problem in frequency-domain terms is when $Z(\infty) + Z'(\infty)$ is singular, but this does not normally cause difficulties in classical synthesis procedures.

The procedure to be described below achieves a synthesis of a network of the form (1), with $(J + J')$ singular, by deriving a new network for which $(J + J')$ is nonsingular. In effect, it is a state-space version of the Foster preamble to classical synthesis techniques [6, ch. 10], in that a number of reactive elements are removed in order to simplify the network. In actual detail the procedure in many ways resembles a multiport Cauer synthesis [2]. The key step is an inversion of the system to be synthesized, so that a zero at infinity becomes a pole at infinity; this is then followed by a reactance extraction.

II. The Synthesis Algorithm

In what follows, it is assumed for simplicity of exposition that u is a vector of port currents, and y is a vector of port voltages. In other words, $Z(s)$ is an impedance matrix. The ideas apply equally well to admittance matrix synthesis (and also, with some modifications, to hybrid matrix synthesis) if one makes the obvious changes such as "series" to "parallel," or "shunt capacitor" to "series inductor." These will not be indicated explicitly. It is important, however, to keep track of whether the ports are assumed to be voltage or current excited. The actual algorithm follows.

1) Gyrator Extraction: Define

$$\hat{J} = \tfrac{1}{2}(J + J')$$

and

$$\hat{y} = H'x + \hat{J}u.$$

Then

$$y = \hat{y} + \tfrac{1}{2}(J - J')u.$$

The second term may be synthesized using transformer-coupled gyrators [1, ch. 8]; these are then placed in series with a new network (to be derived) with input u and output $\hat{y}$. Now drop the superscript hats.

2) Transformation of Port Variables: Find a real orthogonal V such that

$$J = V' \begin{bmatrix} J_1 & 0 \\ 0 & 0_{p_1 \times p_1} \end{bmatrix} V$$

with J_1 nonsingular and symmetric. Define

$$\hat{u} = Vu \qquad \hat{y} = Vy.$$

The new network may be synthesized as the cascade connection of a multiport transformer of turns ratio V, and a new system

$$\dot{x} = Fx + \hat{G}\hat{u}$$
$$\hat{y} = \hat{H}'x + \hat{J}\hat{u}$$

where

$$\hat{G} = GV' \qquad \hat{H} = HV' \qquad \hat{J} = \begin{bmatrix} J_1 & 0 \\ 0 & 0_{p_1 \times p_1} \end{bmatrix}.$$

Now drop the superscript hats.

3) Elimination of Redundant Ports: With p_1 defined as in the last section, partition G as

$$G = [G_1 \quad G_2]$$

where G_2 has p_1 columns. If the columns of G_2 are linearly independent, proceed to the next section. Otherwise, there exists a nonsingular matrix V_2 such that

$$G_2 V_2 = [\hat{G}_2 \quad 0_{n \times p}]$$

where n is the dimension of the F-matrix, and

$$p = p_1 - \operatorname{rank}(G_2).$$

Introduce the transformer (of turns ratio matrix V^{-1}) defined by

$$\hat{y} = V'y \qquad \hat{u} = V^{-1}u$$

where V is defined as

$$V = \begin{bmatrix} I & 0 \\ 0 & V_2 \end{bmatrix}.$$

The new state equations are now

$$\dot{x} = Fx + \hat{G}\hat{u}$$
$$\hat{y} = \hat{H}'x + \hat{J}\hat{u}$$

with $\hat{G} = GV$, $\hat{H} = HV$, and $\hat{J} = V'JV$. Actually, these matrices are of the form

$$\hat{G} = [G_1 \quad G_2 \quad 0_{n \times p}] \qquad \hat{H} = [H_1 \quad H_2 \quad H_3]$$

$$\hat{J} = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0_{p \times p} \end{bmatrix}.$$

A key observation at this point is that the transformations used so far have not changed the passive character of the network. Consequently, the lemma may be applied, and this yields easily

$$H_3 = 0.$$

Let the new port variables be partitioned as

$$\hat{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \hat{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where u_2 and y_2 correspond to the last p-ports. The state equations are now

$$\dot{x} = Fx + [G_1 \quad \hat{G}_2]\, u_1$$

$$y_1 = [H_1 \quad H_2]'x + \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} u_1$$

$$y_2 = 0$$

which implies that the last p-ports may be realized as short circuits. From this point onwards the last p-ports are dropped from consideration; u and y are redefined to be u_1 and y_1. Similarly, the matrices G, H, and J are redefined in the obvious manner.

It may be noted that the last two steps have resulted in the cascade connection of two multiport transformers. The final p output-ports of the second transformer are to be terminated with short circuits, while the remaining output-ports will be terminated with a network yet to be derived. (One could of course combine the results of these two steps to yield a single multiport transformer.)

4) Reactance Extraction: If the new J-matrix is nonsingular, the remainder of the synthesis proceeds as in conventional methods [1]. If, however, J is still singular, then J and G will be of the form

$$J = \begin{bmatrix} J_1 & 0 \\ 0 & 0_{q\times q} \end{bmatrix} \qquad G = [G_1 \quad G_2]$$

where (by virtue of the preceding steps) the q columns of G_2 are linearly independent. There exists therefore a nonsingular matrix T_1 such that

$$T_1 G = \begin{bmatrix} G_a & 0 \\ G_b & I_{q\times q} \end{bmatrix}.$$

Let a new state vector be defined as $\tilde{x} = T_1 x$. Also define $\tilde{F} = T_1 F T_1^{-1}$, $\tilde{G} = T_1 G$, and $\tilde{H} = (T_1^{-1})' H$, so that the state equations are now

$$\dot{\tilde{x}} = \tilde{F}\tilde{x} + \tilde{G}u$$
$$y = \tilde{H}'\tilde{x} + Ju.$$

The lemma may now again be invoked to derive a special property of $\tilde{H}$. With the same partitioning as $\tilde{G}$, $\tilde{H}$ has the form

$$\tilde{H} = \begin{bmatrix} H_a & H_b \\ H_c & H_d \end{bmatrix}$$

where it turns out that H_d is nonsingular (and, in fact, symmetric). It follows that the matrix T_2, defined as

$$T_2 = \begin{bmatrix} I & 0 \\ H_b' & H_d' \end{bmatrix}$$

is invertible.

A further transformation of the state vector may therefore be defined via $\hat{x} = T_2\tilde{x}$. Now let the state vector and both port variables be partitioned as

$$\hat{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

where the partitioning is such as to isolate the last q entries of each vector. With conformable partitioning of the coefficient matrices, the state equations turn out to be of the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} G_{11} & 0 \\ G_{21} & C^{-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} H_{11}' & H_{21}' \\ 0 & I \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} J_1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

where C is a positive definite symmetric matrix.

The crucial step is now to define a new set of port variables

$$\begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} u_2 \\ u_2 - C\dot{x}_2 \end{bmatrix}. \tag{4}$$

The physical significance of this transformation is that the impedance matrix has been inverted (i.e., the roles of u and y have in effect been switched). At the same time, a shunt capacitor C—actually a multiport transformer terminated by capacitors—has been extracted, as shown in Fig. 1. The state equations of the remainder of the network are readily seen to be

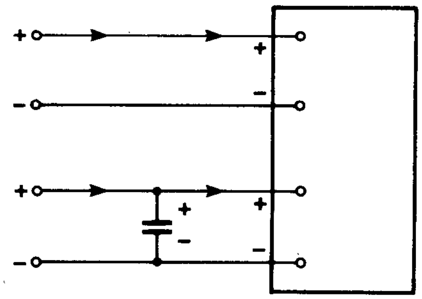

Fig. 1. Reactance extraction.

$$\dot{x}_1 = [F_{11} - G_{11}J_1^{-1}H_{11}']\, x_1 + [G_{11}J_1^{-1} \mid F_{12} - G_{11}J_1^{-1}H_{21}'] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}$$

$$\begin{bmatrix} \hat{y}_1 \\ \hat{y}_2 \end{bmatrix} = \begin{bmatrix} -J_1^{-1}H_{11}' \\ \hline CG_{21}J_1^{-1}H_{11}' - CF_{21} \end{bmatrix} x_1 + \left[\begin{array}{c|c} J_1^{-1} & -J_1^{-1}H_{21}' \\ \hline -CG_{21}J_1^{-1} & CG_{21}J_1^{-1}H_{21}' - CF_{22} \end{array}\right] \begin{bmatrix} \hat{u}_1 \\ \hat{u}_2 \end{bmatrix}.$$

With the appropriate redefinitions of the various matrices, these equations are again of the form

$$\dot{x} = Fx + Gu$$
$$y = H'x + Ju.$$

Moreover, it follows easily from the lemma that the new subsystem may be synthesized using only passive components.

If the new J-matrix is nonsingular, no further transformations need be made. If J is still singular, one simply repeats the entire procedure, starting again from step 1). The process must eventually terminate, since each pass through the "loop" will either cause the number of ports to diminish step 3) or will reduce the dimension of the state vector [step 4)]. Eventually, either one of these dimensions will shrink to zero, yielding a complete synthesis, or a nonsingular J will be encountered. An interesting special case is where the network to be synthesized is lossless. It may readily be verified that in this case the procedure does not terminate until the dimension of the state vector shrinks to zero, so that a complete synthesis is obtained. The resulting network resembles that obtained by the *first Cauer synthesis* [2, ch. 7].

One word of caution is necessary here. If the original network to be synthesized represents an impedance matrix, then (4) will change the problem to that of realizing an admittance matrix. When step 4) is applied a second time, the "shunt capacitors" will in fact become series inductors, and a number of other similar changes will have to be made. This will cause no difficulty, but it does underline the necessity of keeping track of whether a given port variable is currently assumed to represent a current or a voltage.

III. Conclusions

Given a set of state equations representing an electrical network, it is intuitively clear that synthesis will be simple provided that all elements of the state vector represent inductor currents or capacitor voltages. If this condition does not hold, it is desirable to find a transformation of the state equations such that the new state vector satisfies the condition. Such a transformation may easily be found [1] when the solution P of (3) is known; in this paper, the transformation has in effect been found without solution of (3). It is apparent that the methods of this paper work *precisely when (3) is difficult to solve.*

The techniques discussed here may actually be used to solve these equations. To see this, note that step 4) of the algorithm uses a change of variables of the form $\hat{x} = Tx$; it is easily shown that this induces a corresponding change $P = T'\hat{P}T$ in (3). A straightforward calculation then shows that $\hat{P}$ is block diagonal, with one of the blocks a known quantity. Each time step 4) is applied, a further block of P becomes known. When $(J + J')$ ultimately becomes nonsingular, the remaining unknown components of P can be found by solving a reduced-dimensional form of (3).

It is of some interest to note that the synthesis procedure leaves open a number of options. To see this, suppose that $Z(s)$ is the immittance to be synthesized, and that $Z_1(s)$ is the immittance obtained (with $Z_1(\infty)$ nonsingular and symmetric) after the lossless extractions of Section II. Then the following may easily be verified.

1) A synthesis of $Z(s)$ with the minimum possible number of reactive elements follows from a synthesis of $Z_1(s)$ with the minimum possible number of reactive elements.

2) A synthesis of $Z(s)$ with the minimum possible number of resistors follows from a synthesis of $Z_1(s)$ with the minimum possible number of resistors.

3) If $Z(s)$ is symmetric, then so is $Z_1(s)$, and a reciprocal synthesis of $Z(s)$ follows from a reciprocal synthesis of $Z_1(s)$.

In consequence, the procedures of this paper may be used as a preamble to minimal-reactive, minimal-resistive, or reciprocal syntheses, without prejudice to the aims of these syntheses.

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Active *RC* *n*-Port Network Synthesis Using Nullators and Norators

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***Abstract*—A new method of synthesizing active *RC* *n*-port networks using nullators and norators is given. The method based on the reactance extraction principle uses a minimum number of grounded capacitors and gains its importance from the fact that the synthesis procedure does not depend on any topological considerations. A bound on the maximum number of nullator-norator pairs required to realize any arbitrary $Y(s)$ matrix is obtained.**

Manuscript received February 5, 1973; revised May 10, 1973. This work is part of a dissertation submitted to the Indian Institute of Technology, Madras, India, in partial fulfillment of the requirements for the Master of Science degree, by P. A. Ramamoorthy.
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I. Introduction

The use of nullator–norator pairs as the active building blocks in active *RC* network synthesis is well known. As an ideal transistor and an operational amplifier can be approximated by a nullator–norator pair [1], much interest has been shown in analyzing [1]–[4] and synthesizing [5]–[8] with nullators and norators as a prelude to the analysis and synthesis of networks containing resistors, capacitors, and transistors or operational amplifiers. In a recent paper, Yarlagadda and Ye [9] present a method of synthesizing networks with resistors, capacitors, and nullator–norator pairs which heavily depends on