minimal rejection attenuation and was conceived as an inverse Chebyshev filter: its attenuation is plotted by the broken line in Fig. 4. In the passband, the waveguide filter presents
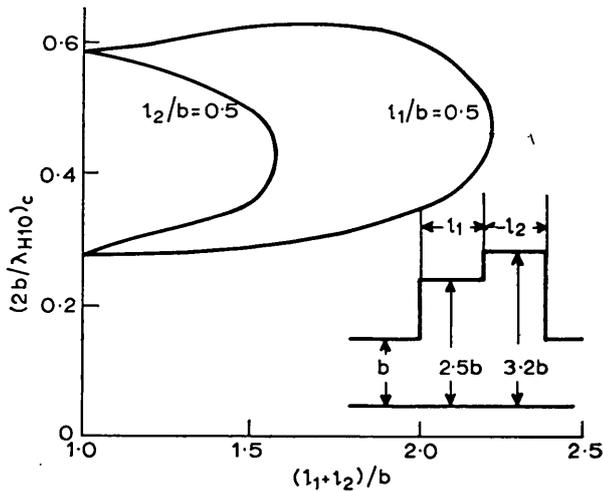

Fig. 3A *Stop frequencies $(2b/\lambda_{H_{10}})_c$ of the H_{10} mode in a stepped E plane stub filter*
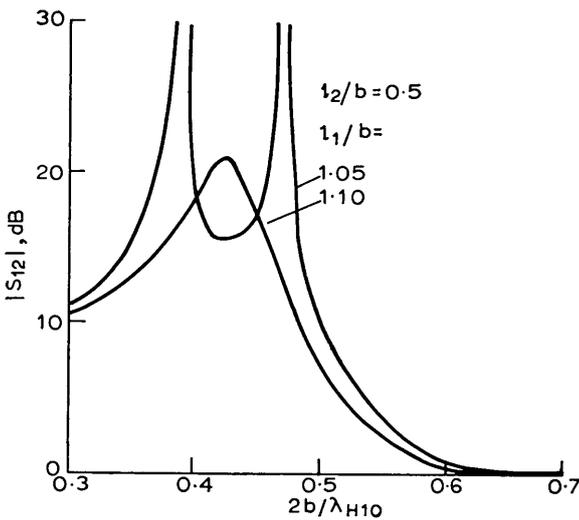

Fig. 3B *Attenuation $|S_{12}|$ of the H_{10} mode in stepped stub filters according to Fig. 3A as a function of the frequency $2b/\lambda_{H_{10}}$*
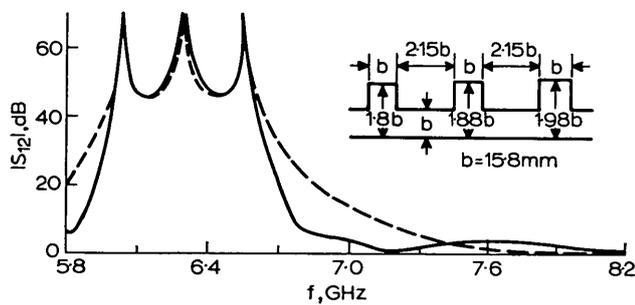

Fig. 4 *Attenuation $|S_{12}|$ of the H_{10} mode in three cascaded stub filters*

The attenuation of a Chebyshev bandstop filter composed of lumped elements is shown by the broken line

an attenuation of less than 4 dB. The stubs are spaced by more than half the guide wavelength of the H_{10} mode at the mid-frequency of the stopband. Higher modes are therefore attenuated by more than 96 dB between the stubs and are disregarded in the numerical calculation.

I am indebted to J. Hagenauer and G. Kowalski for the numerical calculations.

W. BRAECKELMANN   *4th September 1969*

Lehrstuhl Theoretische Elektrotechnik
Technische Hochschule Darmstadt
61 Darmstadt, W. Germany

502

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OPTIMAL REGULATOR WITH BOUNDS ON THE DERIVATIVE OF THE INPUT*

The problem of minimising $\int_{t_0}^{\infty} (x'Qx + u'Ru)dt$ for a completely controllable linear system, subject to the constraint $|\dot{u}| \leqslant 1$, is shown to be equivalent to a singular optimal-control problem; the known singular theory is then used to obtain the optimal control.

The standard regulator problem of optimal control is to minimise the performance index $V[x(t_0), u, t_0] = \int_{t_0}^{\infty} (x'Qx + u'Ru)dt$ for the linear dynamical system $\dot{x} = Fx + Gu$. F and G may possibly be time varying, and the control u may be either a scalar (single-input system) or a vector (multiple-input system).

If no constraints are imposed on u, the problem is simply the standard regulator problem for which a solution has been known for some years.[1] The regulator problem with the additional constraint $|u| \leqslant 1$ (the 'problem of Letov') has recently been solved; see, for example, Reference 2. However, to the authors' knowledge, no consideration has been given to the regulator problem with the restriction $|\dot{u}| \leqslant 1$, which would limit rapid changes in the control. A little thought will show that there are, in fact, practical applications which require this restriction. This, then, is the problem considered in this letter.

The usual assumptions will be made concerning the system, namely that the pair $[F, G]$ is completely controllable for all $t \geqslant t_0$, and Q and R are nonnegative definite and positive definite, respectively. For clarity of notation, u will be considered to be a scalar, but the results given here are equally applicable to multiple-input systems.

To make the problem tractable, we introduce the new variables

$$\hat{u} = \dot{u} \text{ and } \hat{x} = \begin{bmatrix} x \\ u \end{bmatrix}$$

[where the initial condition $u(t_0)$ will be specified later]. This gives the new system shown in Fig. 1. The state equation of the new system is

$$\dot{\hat{x}} = \hat{F}\hat{x} + \hat{G}\hat{u} \text{ where } \hat{F} = \begin{bmatrix} F & G \\ 0 & 0 \end{bmatrix} \qquad \hat{G} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the performance index becomes

$$V = \int_{t_0}^{\infty} \hat{x}'\hat{Q}\hat{x}dt \text{ where } \hat{Q} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}$$

The control-variable constraint is now $|\hat{u}| \leqslant 1$.

The new problem is now the 'singular' optimal-control problem, solved for the time-invariant single-input case in
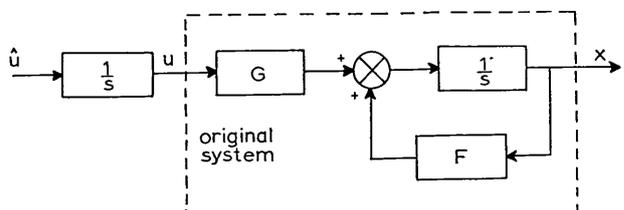

Fig. 1 *Modified system*

References 3 and 4 and for the time-varying multiple-input case in Reference 5. The optimal $\hat{u}$ turns out to be 'bang–bang' almost everywhere, but of the form $\hat{u} = K'\hat{x}$ on the 'singular strip' $K_1'x = 0$, $|K'\hat{x}| \leqslant 1$, where K and K_1 will be given below. Moreover, the optimal bang–bang control law near the origin (excluding the singular strip) is given by

$$\hat{u} = \text{sgn}\,(K_1'\hat{x})$$

For large $\hat{x}$, this law is no longer optimal, but it is probably the best practical approximation to the true optimal bang–bang law (which is, in general, a rather complicated function of $\hat{x}$).

In practice, the dual-mode control proposed above—bang–bang in some regions, linear in others—would probably be inconvenient. However, Reference 6 has shown that, if a bang–bang law such as that proposed is used throughout, all trajectories reaching the singular strip will chatter or slide along it in such a way that the effective control law is the linear one above. A practical realisation of the optimal control is then simply $\hat{u} = \text{sgn}\,(K_1'\hat{x})$, which is strictly optimal for small x, and near optimal for large x.

To compute K_1, we use the following procedure (described more fully in References 4 and 5): Form the matrices

$$G_1 = \hat{F}\hat{G} - \dot{\hat{G}} \quad S_1 = \hat{Q}\hat{G} \quad R_1 = \hat{G}'\hat{Q}\hat{G}$$

and let $\Pi(t, t_1)$ be the solution of the equation

$$-\dot{\Pi} = \Pi(\hat{F} - G_1 R_1^{-1} S_1') + (\hat{F}' - S_1 R_1^{-1} G_1')\Pi$$
$$-\Pi G_1 R_1^{-1} G_1'\Pi - S_1 R_1^{-1} S_1' + \hat{Q} \text{ with } \Pi(t_1, t_1) = 0$$

Now put

$$P(t) = \lim_{t_1 \to \infty} \Pi(t, t_1)$$

The assumptions made concerning Q and R and the complete controllability of $[F, G]$ ensure that this limit will, in fact, exist. Then

$$K_1 = -(PG_1 + S_1)R_1^{-1}$$

Also, the matrix K (if it is required) is given by

$$K = \dot{K}_{11} + \hat{F}'K_1$$

In the present case, we can partition P as

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2' & P_3 \end{bmatrix}$$

where the partitioning scheme is the same as that of $\hat{F}$ or $\hat{Q}$. Substituting all partitioned matrices into the above Riccati differential equation, it is found that

$$-\dot{P}_1 = P_1 F + F'P_1 - P_1 GR^{-1}G'P_1 + Q \quad P_2 = 0$$
$$P_3 = 0$$

Also

$$K_1 = -(PG_1 + S_1)R_1^{-1} = \begin{bmatrix} -P_1 GR^{-1} \\ -1 \end{bmatrix}$$

The equation of the singular strip is $K_1'\hat{x} = R^{-1}G'P_1 x + u = 0$, and, if we set $u(t_0) = -R^{-1}G'P_1 x(t_0)$, and $x(t_0)$ is sufficiently small for $|\hat{u}| < 1$ to hold, the system trajectory will always stay on the singular strip, and u will always be given by $u = -R^{-1}G'P_1 x$.

This, as it turns out, is precisely the optimal control for no constraints on $\hat{u}$. This is an expected result, since the singular

strip is merely that region where the control lies within its constraints. The significance of this result is that, if we are free to specify $u(t_0)$, we should choose it so that the initial point in (x, u) space is on the singular strip; for a given $x(t_0)$, any other u will certainly be 'less optimal' than a result which was derived without assuming any constraints on u.

For large $x(t_0)$, it is, in general, impossible to choose a $u(t_0)$ so that the initial point is on the singular strip. The optimal (or, strictly speaking, the suboptimal best practical approximation to the optimal) u is then given by a bang–bang law; i.e.

$$\hat{u} = \text{sgn}\,K_1'\hat{x}$$

or $\quad \hat{u} = -\text{sgn}\,(R^{-1}G'P_1 x + u)$

This gives the feedback configuration of Fig. 2. If we are not free to choose $u(t_0)$, this completes the solution.

Sufficient conditions for the stability of the closed-loop system may be found in References 3–5. In general, it is difficult to guarantee global asymptotic stability, but the system may always be made asymptotically stable in a neighbourhood of the singular strip by appropriate choice of Q.

P. J. MOYLAN   *29th August 1969*
J. B. MOORE

Department of Electrical Engineering
University of Newcastle
NSW 2308, Australia

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ELIMINATION OF DELAY TIMES IN XBAND GUNN-EFFECT OSCILLATORS USING R.F. INJECTION

A simple theory is given showing how a small amount of r.f. power injected at a frequency near the Gunn-diode frequency can eliminate delay times in pulse operated Xband Gunn-effect oscillators.

Introduction: In a recent paper,[1] the author discussed theoretically the nature of growing oscillations in Gunn-effect oscillators. In particular, results were obtained for the delay times associated with pulse-operated Xband Gunn-effect oscillators.

If the bias voltage is very large compared with the threshold voltage, then, on satisfying certain relations between the low-field parameters of the Gunn diode and the equivalent *LCR* parameters of the microwave cavity, a minimisation of the delay times could be achieved. If the bias voltage is made smaller, e.g. 2 to 3 times threshold, the delay time will increase often by many orders of magnitude.

In this letter it is shown that, for low-bias conditions, by injecting a small amount of r.f. power into the microwave cavity (see Fig. 1), the delay times can be totally eliminated.

Notation: The notation and specification of delay times are identical to those used in Reference 1. The differential equation is defined by eqn. 35 of Reference 1.

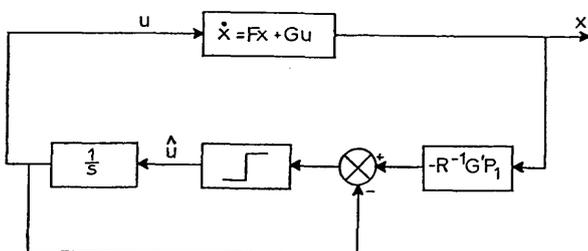The differential equation describing the nature of growing

Fig. 2 *Feedback configuration*