The condition for a supply rate to be interesting

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ABSTRACT: Some results in dissipative systems theory require that the supply rate be such that it can be forced negative by choice of input. In an earlier paper [Moy18] an eigenvalue condition was shown to be sufficient for this to be true. It now turns out that that condition is also necessary.

1. Introduction

The theory of dissipative systems [Wil72] [HM80] [Moy14] defines a system to be dissipative if a certain quantity that depends on the input and output is nonnegative for all possible inputs For time-invariant continuous-time systems this is normally expressed in terms of an integral:

> **Definition 1**. A system with input u and output y is dissipative, in the input-output sense, with respect to supply rate w iff
> $$\int_0^T w(u(t), y(t)dt \geq 0$$
> for all $T\geq0$ and all inputs u.

The supply rate is a function of two variables Suppose that it has the property that $w(u,y)\geq0$ for all u and y. In that case, the above inequality will always be satisfied, so that *every* system is dissipative with respect to that supply rate. This is not a useful case. To make dissipativeness useful, we need to permit the supply rate to go negative, so that the inequality defines a property of the system rather than simply of w.

Because of this, a class of "interesting" supply rates was introduced in [HM80].

> **Property A**. For any $y\neq0$, there exists a $u(y)$ such that $w(u(y),y)<0$.

Property A has proven to be useful in establishing several results. In [HM80], it turned out to be the main condition needed to ensure that an internal storage function is positive definite. In [Moy75] it was shown to be a condition that allowed establishing a link between time-domain and frequency-domain criteria in linear-quadratic control theory. In [Moy14, chapter 8], which deals with frequency domain conditions for dissipativeness, it turns out to be the assumption needed to link behaviour on the $j\omega$ axis to behaviour elsewhere on the complex plane.

In this paper we are concerned with the case where the supply rate is a quadratic function of its arguments

$$w(u, y) = y^T Qy + 2y^T Su + u^T Ru = [y^T \quad u^T] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix}$$

For clarity, then, we should probably restate Property A.

> **Property B**. For any $y\neq0$, there exists a $u(y)$ such that $[y^T \quad u^T] \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} < 0$.

2. The previous result

Let u be an m-vector and y an n-vector, and define

$$M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$$

Then the following result was shown in [Moy18].

Theorem 1. A sufficient condition for Property B to hold is that M have at least n negative eigenvalues.

At the time, it was unclear whether an equally simple necessary condition could be found.

3. Partitioning a finite-dimensional space

Let M be a $p \times p$ matrix. Because it is real and symmetric, all of its eigenvalues are real, and its eigenvectors can be chosen[1] to be mutually orthogonal. Suppose that it has p_1 negative eigenvalues, and let x_i be the corresponding eigenvectors, for $i = 1..p_1$, Those vectors form a p_1-dimensional subspace. Any vector in that subspace can be written as the sum

$$x = \sum \alpha_i x_i$$

for some scalars α_i. Now we have

$$Mx = \sum \alpha_i M x_i = \sum \alpha_i \lambda_i x_i$$

$$x^T M x = \sum \alpha_j x_j^T \sum \alpha_i \lambda_i x_i = \sum \lambda_i \alpha_i^2 x_i^T x_i$$

where the final equality comes from the fact that $x_j^T x_i = 0$ if $j \neq i$. Finally, we conclude that $x^T M x < 0$ for all x in the space defined by the negative eigenvalues and their corresponding eigenvalues.

By the same argument, we have $x^T M x = 0$ for all x in the p_2-dimensional subspace defined by the eigenvectors corresponding to the zero eigenvalues, if any; and $x^T M x > 0$ for all x in the remaining p_3-dimensional subspace, with $p_1 + p_2 + p_3 = p$. This leads to the following result.

Lemma 1. The set of all x such that $x^T M x < 0$ is precisely the space spanned by the eigenvectors of M corresponding to its negative eigenvalues.

Proof. Let S^-, S^0, and S^+ be the subspaces defined by the eigenvectors corresponding to the negative, zero, and positive eigenvalues, respectively. We have just shown that $x^T M x < 0$ for all $x \epsilon S^-$, with corresponding results for the other two subspaces But, by the properties of eigenvectors, those three subspaces cover the entire p-dimensional space, with no overlap. (S^0 is e closed subspace, even though it is unbounded, while S^- and S^+ are open spaces, so there is no question of their overlapping at boundaries of the sets.) Clearly any x such that $x^T M x < 0$ cannot be in S^0 or S^+, so it must be in S^-.

This result has the following corollary.

[1] If two or more eigenvalues are equal, the corresponding eigenvectors are not uniquely defined. In this case it turns out to be possible to choose, among the possibilities, eigenvectors that are orthogonal to one another.

Lemma 2. Let $x_1, x_2, \ldots$ be a set of vectors such that $x_i^T M x_i < 0$ for all x_i in the set. If the matrix $[x_1 \quad x_2 \quad \cdots]$ has rank k, then M has at least k negative eigenvalues.

Proof. The space spanned by those vectors has dimension k, so p_1 is at least as large as k, where p_1 is the number of eigenvectors of M corresponding to its negative eigenvalues.

4. The main result

We are now able to state the final result.

Theorem 2. Let n be the size of the output vector y. Then Property B holds iff the matrix

$$M = \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix}$$

has at least n negative eigenvalues.

Proof. One half of this theorem was proved in [Moy18]. For the other half, let $\{y_i\}$ be a set of n unit vectors. If Property B holds, then for each of those y_i there exists a u_i such that $w(u_i, y_i) < 0$. Let

$$x_i = \begin{bmatrix} y_i \\ u_i \end{bmatrix}$$

and form the matrix

$$[x_1 \quad x_2 \quad \cdots \quad x_n] = \begin{bmatrix} y_1 & y_2 & \cdots & y_n \\ u_1 & u_2 & \cdots & u_n \end{bmatrix}$$

We know nothing about the linear independence, or otherwise, of the u_i, but this does not matter. By construction, the top n rows of the overall matrix form a unit matrix, so the matrix we have constructed has rank n. From Lemma 2, the matrix M must have at least n negative eigenvalues.

5. Conclusions

Theorem 2 fills a hole in the theory of dissipative systems. We have known for a long time that Property A, and its more specific form Property B, are important to proofs of some results, and that it must be related in some way to the matrix that we here call M, but until now it was not clear that an eigenvalue condition could be a *necessary and sufficient* condition for the desired property.

6. References

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